# MASARYKOVA UNIVERZITA Přírodovědecká fakulta

ÚSTAV TEORETICKÉ FYZIKY A ASTROFYZIKY

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### MASARYKOVA UNIVERZITA Přírodovědecká fakulta Ústav teoretické fyziky a astrofyziky

# Limity inverzní metody pro určování rotace a tvaru asteroidů

Diplomová práce **Vendula Slavíková** 

Vedoucí práce: Mgr. Peter Scheirich, Ph.D. Brno 2023

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### Abstrakt

V této diplomové práci se zaměřujeme na zkoumání limitů metody inverze světelných křivek. Snažíme se najít takové požadavky kladené na fotometrická data, aby metoda vracela co nejjednoznačnější výsledek. Fotometrická data, parametry modelů planetek a programy provádějící inverzní metodu byly získány z databáze DAMIT. Byly vypočteny tepelné mapy pro různá nastavení souřadnic rotační osy a v nich vynesen RMS rozdíl mezi vstupními a výstupními světelnými křivkami. Spočítali jsme rotační póly pro 4 objekty a porovnali naše výsledky s hodnotami z databáze. Zkoumali jsme také vliv použití upravených vstupních dat na tepelné mapy. Bylo testováno snížení počtu vstupních pozorování a s tím související lokalizovatelnost rotačního pólu. Hledali jsme závislost mezi počtem vstupních relací použitých pro úspěšně lokalizovaný ekliptikální pól a prostorovým rozložením těchto pozorování. Podobnou závislost jsme hledali i pro časové rozložení. Rovnoměrnost rozdělení vykreslených fází periody pro každou sadu pozorování byla vyhodnocena pomocí Kolmogorovoa-Smirnovoa testu.

### Abstract

In this thesis, we focus on investigating the limitations of the light curve inversion method. We attempt to find the requirements of the photometric data for the method to return a result as unambiguous as possible. Photometric data, asteroid model parameters, and the inversion method procedures were obtained from the DAMIT database. Heatmaps for different settings of the coordinates of the rotational axis were computed, with the RMS difference between the input and output light curves plotted. The ecliptic poles for 4 objects were reproduced and compared with the database values. We also studied the effects of using modified input data on the heatmaps. Decreasing the number of input observations and the associated localizability of the ecliptic pole was examined. We searched for a dependency between the number of input sessions used for the successfully located ecliptic pole and the spatial distribution of such observations. The same search was conducted for the time distribution. The uniformity of the phase distribution for each observation set was evaluated using the Kolmogorov–Smirnov test.

# ZADÁNÍ DIPLOMOVÉ PRÁCE

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#### Oficiální zadání:

Inverzní metoda pro určování rotačního stavu a tvaru asteroidů z jejich fotometrických dat je již několik desetiletí dobře rozpracována a používána. Přesto má řadu nedostatků, které se projevují zejména tehdy, pokud mají napozorovaná data nedostatečnou kvalitu či kvantitu. V takovém případě metoda dává nejednoznačné výsledky, případně výsledky zatížené velkou neurčitostí. Student se pokusí poněkud vágní pojem "nedostatečná kvalita či kvantita dat" specifikovat. Pokusí se nalézt požadavky, které musí data splňovat, aby metoda dávala jednoznačný výsledek, a dále určit, jak závisí neurčitost výsledku na vlastnostech dat.

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# Poděkování

Na tomto místě bych chtěla poděkovat všem, kteří mě v průběhu psaní práce povzbuzovali nebo podali pomocnou ruku. Rodině, přátelům, partnerovi – bez vaší opory by to nešlo. Mé veliké díky patří Mgr. Martinu Friákovi, Ph.D. za čas věnovaný konzultacím, jeho ochotný přístup a neocenitelné rady. V neposlední řadě bych ráda poděkovala těm, kteří tuto bakalářskou práci četli a pomáhali mi s její stylistickou stránkou.

# Prohlášení

Prohlašuji, že jsem svoji diplomovou práci vypracovala samostatně pod vedením vedoucího práce s využitím informačních zdrojů, které jsou v práci citovány.

Vendula Slavíková

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## Introduction

Asteroids have an important role in modelling the formation and evolution of the Solar System. Their light curves are observed for decades. If the light curves from several observations have different observational geometries, these observations contain quite a bit of information about the rotation state and the shape of the asteroid. These physical parameters can then be determined using the light curve inversion method, which was developed by Kaasalainen et al. (2001) and Kaasalainen & Torppa (2001). This method has been in constant use since then, its reliability has been proven, and thousands of asteroid models have been created using it.

This thesis intends to study the limitations of the inverse method for getting the rotational state and shape of asteroids. These limits include insufficient data quality or quantity. The thesis attempts to find a specific parameter to determine whether the input data are sufficient. We also address the accuracy of the search for the asteroid's rotation pole and the distribution of the different observation geometries.

The first chapter deals with an introduction to the small bodies of the solar system, their formation and subdivision. The second chapter focuses on the characterization of asteroids and how to describe their shape and orbital parameters. The third chapter introduces asteroid photometry and the methods used to obtain shape and rotational state from photometric data. Chapters 4 and 5 are devoted to the actual analysis of the data of the selected objects and the examination of the effects of various data properties on the results.

# **1** Small Solar System Bodies

The Solar System is a gravitationally bound system consisting of the Sun and the objects that orbit it, such as the eight planets, their natural satellites, and small Solar System bodies. According to an IAU resolution [e1], a small Solar System body (SSSB) is an object which is neither a planet, nor a dwarf planet, nor a satellite. SSSBs include interplanetary dust, most of the Solar System asteroids, most Trans-Neptunian Objects, comets, and other small bodies.

From the properties of the objects we observe today, we can determine the conditions of the nebula cloud the Sun was formed from. Therefore, knowledge of the composition of SSSB's gives us information about this original cloud of interstellar matter.

#### **1.1** Formation of the Solar System

The formation of the Solar System started with the gravitational collapse of a cloud of interstellar matter. This cloud consisted mainly of hydrogen gas. According to McCrea (1960), probably about 90 per cent of the cloud's mass was molecular hydrogen. Such gas cloud can contract due to its gravitational instability after meeting the necessary conditions, such as the Jeans criterium. That means the mass of the cloud gets over its Jeans mass  $M_J$ 

$$M_{\rm J} \simeq const. \left(\frac{kT}{G\mu m_{\rm u}}\right)^{3/2} \frac{1}{\sqrt{\rho}},$$
 (1.1)

where k is Boltzmann constant, T and  $\rho$  are temperature and density of the interstellar cloud, G is the gravitational constant and  $\mu$  is average molecule mass in Unified atomic mass units  $m_{\rm u}$ . A collapse of a cloud can be also triggered by an explosion of a nearby supernova star or by a passing density wave.

According to the virial theorem

$$2 \cdot \left\langle E_{\mathbf{k}} \right\rangle + \left\langle E_{\mathbf{p}} \right\rangle = 0 , \qquad (1.2)$$

where  $E_k$  is kinetic energy and  $E_p$  is potential energy, half of the potential energy released by the collapse is transferred into infrared radiation and half into heat. The central part of the collapsing cloud starts to get hotter and denser. Because of the conservation of angular momentum, with a decreasing radius, the protostar rotates faster. Eventually, hydrostatic equilibrium is established, and the central part of the cloud becomes a protostar. As the protostar keeps collapsing, it gets impervious to its radiation, which results in an even higher temperature. When the temperature reaches a few million Kelvin, thermonuclear reactions are ignited. This new source of radiation provides the new star with enough radiation to resist the gravitational force. Therefore, the collapse is over, and the star becomes a Main Sequence Star<sup>1</sup>.

While the protostar is forming, the outer parts of the original cloud are getting flattened to conserve the total angular momentum. A protoplanetary disc is formed. As the protostar rotates very fast, most of the star's angular momentum is transferred to the protoplanetary disc. Since the mass of the disc does not meet the Jeans criterium, the planets could not have been formed by gravitational collapse. The material in the protoplanetary disc starts to accrete into bigger bodies due to collisions, gravitational force and friction. Molecules create clusters, which grow into macroscopic grains. These grains are highly porous, which makes it easy to pick up smaller particles that move slowly relative to them. On the other hand, they are fragile and can fall apart after collisions with fast particles. With their gradual growth, they stick together to form large bodies, eventually measuring up to 1 km in size. We call these bodies planetesimals. This process lasts thousands to hundreds of thousand years.

Planetesimals orbit the Proto-Sun on circular trajectories, and their velocities relative to one another are small. When they collide, they are more likely to stick together than to break. Some of the planetesimals become planetary embryos, depending on their location and surroundings.

Once embryos overcome certain mass, they gravitationally attract dust and gas around them, and this way they gain even more mass. If the mass of the gas envelope gets approximately the same as the mass of the core it contains, the planet very quickly attracts a large amount of gas from its surroundings. This gas has to accrete even more in order to create a compact planet. According to the virial theorem (equation 1.2), the planet has to radiate half of the potential energy out. We can observe this radiation from the gas giant planets in the infrared region to this day. They can also capture smaller planetesimals in their gravitational force field. These smaller objects then become moons.

Terrestrial planets were probably formed by gradual collisions with tens of planetesimals. Due to close encounters of the big planets with each other, the Solar System looked different when it was formed - since then the planets migrated. This migration caused a large amount of the smaller planetesimals to dramatically change their orbit, which resulted in a period called The Late Heavy Bombardment<sup>2</sup>. Terrestrial planets and moons of the giant planets were hit by many of these planetesimals, which caused:

- their differentiation<sup>3</sup>,
- random orientation of their rotational axis,
- transport of water to the terrestrial planets,

<sup>&</sup>lt;sup>1</sup>A Main Sequence Star is a star that is fusing hydrogen. The Main Sequence is a curve on plots of brightness depending on temperature, around which stars seem to be clustering. Stars spend most of their active life as Main Sequence Stars.

<sup>&</sup>lt;sup>2</sup>The theory of The Late Heavy Bombardment started with studies of the origin of the Moon's craters (Baldwin, 1942) and was accepted after radiometric dating of the material collected by Apollo astronauts, performed at the same time at Sheffield University (Turner et al., 1973) and Caltech (Nyquist et al., 1973).

<sup>&</sup>lt;sup>3</sup>Differentiation is a process, in which the material of the planet was recast. Heavier elements, such as nickel or iron, sank to the core, whereas the lighter ones, aluminium compounds, oxygen compounds, silicate compounds, floated at the surface. All terrestrial planets are differentiated.

- creation of first atmospheres,
- creation of Moon<sup>4</sup>,
- and separation of Mercury's silicate mantle<sup>5</sup>.

#### **1.2** Minor planets

Minor planets are the most numerous group of bodies in the Solar System. According to the IAU's Minor Planet Center [e2], there are over 1.1 million discovered minor planets. The first discovered minor planet was (1) Ceres<sup>6</sup>, detected by Giuseppe Piazzi on 1st January 1801.

In The Solar System minor planets are located mostly in the area called Main Asteroid Belt<sup>7</sup>. Some minor planets are orbiting on trajectories that are crossing orbits of planets. Shape and orbital parameters of asteroids and their photometry are further described in chapters 2 and 3.

There are groups of minor planets that show similar properties. They have approximately the same distance from the Sun, the same orbital period, eccentricity and inclination. These groups are called asteroid families. Asteroids in a family origin from one parent body destroyed in a collision.

It is important to closely study different types of minor planets to understand their compositions, structures, sizes, and trajectories. Small fragments of minor planets often hit the Earth in the form of meteorites, larger rocks can however put our civilization in danger. Knowledge of their composition gives us information on how the Solar system was formed, and in future, minor planets can serve as a source of raw materials. As of today, there have been already several landing and sample missions on minor planets, namely the first minor planet landing – (433) Eros (Veverka et al., 2000), the first minor planet sample mission – (25143) Itokawa (Fujiwara et al., 2006), (162173) Ryugu (Watanabe et al., 2019) and (101955) Bennu (Lauretta et al., 2018).

#### **1.3 Trans-Neptunian objects**

A trans-Neptunian object (TNO) is any dwarf or minor planet in the Solar System that moves behind the orbit of Neptune. Not every TNO is a minor planet. In 1930, Pluto was the first TNO discovered. The second TNO (orbiting the Sun directly) was (15760) Albion as late as 1992.

<sup>&</sup>lt;sup>4</sup>The origin of the Moon is explained by The Great Impact Theory. According to this hypothesis, there was a collision of the arising Earth with a protoplanet the size of Mars, which we call Theia. This collision caused the material to be ejected from Earth and Theia into the early Earth's orbit. The material formed a ring, which gradually accreted and developed into our Moon.

<sup>&</sup>lt;sup>5</sup>This explains why nowadays Mercury has a very large core in comparison to its size.

<sup>&</sup>lt;sup>6</sup>At first, Ceres was thought to be a planet. Later on, it was classified as a minor planet. The IAU resolution [e1] from 2006 changed its classification again and Ceres is now considered a dwarf planet. However, it is still listed in the official catalogue of SSSBs under catalogue number 1.

<sup>&</sup>lt;sup>7</sup>Minor planets of the inner Solar System are usually referred to as asteroids.

TNOs are classified based on their distance from the Sun and their orbital parameters. The Kuiper belt objects (KBOs) form a circumstellar disc similar to the Main Asteroid Belt, but much larger. It's located between 30 and 55 au from the Sun. With Pluto and (15760) Albion, Makemake is also one of the KBOs. KBO objects can be further divided into classical KBOs and resonant KBOs locked in an orbital resonance with Neptune, where classical KBOs have almost circular orbits with a small inclination from the ecliptic.

Further away from the Sun there are objects with eccentric and inclined orbits. Their orbits do not cross nor resonate with any other planet's orbit. We call these TNOs the scattered disc objects (SDOs). Eris, the most massive known TNO, is one of them.

#### 1.4 Comets

A comet is a Small Solar System Body. It is composed mostly of ice and dust. When a comet passes close to the Sun, materials within the comet vaporize and create tails. A dust tail is left behind in the comet's orbit, often forming a curve. Gas tail always faces away from the Sun, gas particles affected by the solar wind.

Usually, its orbit is very eccentric. Most comets stay behind Pluto's orbit, from where only occasionally some arrive in the inner Solar System. Gravitational interaction with planets can change a comet's orbit to a less eccentric one or a hyperbolic one, causing it to leave the Solar System.



Figure 1.1: An empirical classification of small solar system bodies. On the vertical axis, there is the object's morphology, on the horizontal axis, Tisserand dynamical parameter  $T_J^8$  is plotted. LPC stands for long-period comets, HFC for Halley family comets, and JFC for Jupiter family comets. The lower left quadrant features likely defunct comets with no activity. Active asteroids from the upper right quadrant are asteroids, that show some comet-like activity – they eject dust or produce comae or tails. Adapted from Jewitt (2012).

<sup>&</sup>lt;sup>8</sup>Tisserand dynamical parameter measured with respect to Jupiter is calculated using the semimajor axis, eccentricity, and inclination of the object's orbit. It provides a measure of the close-approach speed to Jupiter.  $T_J$  for Jupiter is equal to 3.

The differences between a comet and a minor planet are based on two parameters - orbital properties and physical appearance. As seen in Figure 1.1, in the upper left quadrant we can find traditional comets, in the lower right asteroids.

We divide comets into short-period ones and long-period ones, depending on their orbital periods. Long-period comets are believed to come from the Oort cloud, they have high eccentricities. Comets with periods shorter than 200 years are generally defined as short-period comets.

# 2 Characterization of asteroids

An asteroid with a determined orbit gets labelled by a number in brackets and a name. The number corresponds with the order in which this asteroid was discovered. The name has to be approved by IAU first. The scientist that contributed to the determination of the orbit the most is usually the one naming the asteroid.

#### 2.1 Shape and composition

Asteroids come in various shapes. Some are elongated, some nearly spherical. Most of them are of irregular shape. Scientists are finding more and more so-called doubles – a system of two asteroids of similar sizes orbiting each other, sometimes even touching. Most asteroids with sizes between 200 m and 10 km are assumed to be composed of rocks joined by their gravity – we call these rubble piles asteroids (Walsh, 2018).

Although some asteroids were observed directly, their shapes are mainly determined by observing their changing brightness. More about this topic is described in chapter 3. Real shapes of the asteroids are difficult to use during modelling. That is why convex polyhedrons with triangular surface facets are used instead. Even though these polyhedrons are only approximations, it has been proven to be good enough to fit the light curves and to derive the asteroid's main physical parameters.

Not counting in (1) Ceres, which is now classified as a dwarf planet, (4) Vesta is the most massive asteroid in the Main Asteroid Belt<sup>9</sup>. Therefore, the size of an asteroid can be as much as a little over 500 km. The majority of them are in fact smaller than that. It has been estimated that about 30 asteroids are larger than 200 km, and 250 asteroids are larger than 100 km [e3]. The total mass of all the asteroids combined is less than the mass of the Moon.

We can estimate the composition of an asteroid from its albedo, spectrum and density. Density is usually estimated by observing the moons the asteroid might have. However, for most of the bodies, it is unknown. Chapman et al. (1975) has divided asteroids based on their spectroscopy into three groups - C, S and U, with U being a group for all the asteroids that did not fit C or S.

C-type (carbonaceous) asteroids have low albedos. Along with carbon, they are composed of rocks and minerals. They are the most common type – they take up to 75 % of all asteroids. The spectrum of a C-type asteroid is very similar to the one of

<sup>&</sup>lt;sup>9</sup>It might also be the largest one in diameter, but it overlaps with the uncertainty of (2) Pallas, which is now considered to be larger.

carbonaceous chondrite meteorites. That means their chemical composition is very close to the original cloud the Solar System was formed from.

- Approximately 17 % of asteroids are S-type (siliceous) asteroids. Their spectra suggest they have siliceous mineral composition S-type asteroids are similar to stony meteorites. They have a relatively high density and a moderate albedo.
- Nowadays, we also have M-type asteroids<sup>10</sup>. These objects are metallic they contain nickel and iron. They are assumed to be the source of iron meteorites.

Distance from the Sun in which different asteroids have formed is related to the differences between the composition of individual groups. C-type asteroids are located mostly at the outlying edge of the Main Asteroid Belt. The inner parts of the Belt are where the S-type asteroids prevail.

#### 2.2 Orbital parameters

Around 95 per cent of catalogued minor planets are located between 2.1 and 3.3 au – the Main Belt. There are about 1600 Near-Earth asteroids that come near Earth and might become a threat in future. Another group of asteroids share the same orbit as a planet, staying near Lagrangian points L4 and L5, where the gravitational pulls from the Sun and the planet are balanced. These asteroids are called Trojans, and although Jupiter's ones are the most famous, it is not the only planet having them.

As all SSSBs are moving on orbits that are conic sections, most asteroids have elliptical orbits with the Sun in one of the foci. The motion of any cosmic body can be characterized by orbital elements - eccentricity, inclination, semi-major axis, the longitude of the ascending node, the argument of periapsis (perihelion in the Solar System), and mean anomaly, with their respective meanings explained in table 2.1.

To describe the exact position of an asteroid in its orbit, an ecliptic coordinate system is used (figure 2.3). Coordinates and uniquely determine the position of the asteroid relative to the ecliptic. It is also possible to use the astrocentric cartesian coordinate system (figure 2.2). The origin of this system lies in the asteroid. At every time, there are two vectors - one towards the Sun, the other towards the Earth. In total, this gives us six coordinates that also fully describe the position of an object.

Speaking about rotation, we can divide asteroids of the Main Belt and the Near-Earth ones into fast, normal and slow rotators. Fast rotators spin faster than 11 revolutions per day, normal and slow rotators are slower than that. Looking at the figure 2.4, it is noticeable that there are almost no big fast rotators. On the other hand, there are numerous groups of big slow rotators and small fast rotators.

<sup>&</sup>lt;sup>10</sup>In the Tholen classification, the M-type asteroids are part of the X-group. Along with M-type, there are also E-type and P-type asteroids. Both are spectrally indistinguishable from M-type. E-type asteroids have high albedos and are found closer to the Sun than C-type. P-type are similar to C-type asteroids, though have lower densities than C-type. Albedo is the main difference between E, M and P types. When there is no information about the albedo, the object is classified as an X-type object.

Orbital element	Explanation	Sign		
Eccentricity	Describes how much the orbit differs from a perfect			
	circle.			
Inclination	The angle between the orbit and the ecliptical plane.	i		
Semi-major axis	The longest diameter of an ellipse, half of the distance	a		
	between perihelion and aphelion.			
Longitude of the as-	The angle between the position vector of the ascend-	Ω		
cending node	ing node and the reference direction of the coordinate			
	system (the vernal equinox in the ecliptic coordinate			
	system).			
Argument of perihelion	The angle between the position vector of the ascending	M		
	node and the position vector of the perihelion.			
Mean anomaly	The angular distance from the perihelion that a ficti-	ω		
	tious body would have if it moved with constant speed			
	in a circular orbit of a diameter equal to the semi-major			
	axis of the actual orbit.			

Table 2.1: A table with orbital elements used to describe an orbit of a cosmic body moving in the Solar System.

For the orientation of a rotational axis, an ecliptic pole is used – the ecliptic coordinates of the unit vector of said axis. That works for a stationary axis. Unfortunately, an asteroid does not have to have only one axis that it rotates around.

As many asteroids are only rubble piles, their rotational velocity is limited. The centripetal acceleration can not surpass gravitational acceleration, otherwise the asteroid would not hold together. For a hypothetic asteroid consisting of a large spherical body and a small rubble object, we can calculate the limit rotational velocity this system can withhold as follows

$$\frac{GMm}{R} = \frac{1}{2}mv^2 , \qquad (2.3)$$

where R is the radius of our hypothetic asteroid, G is gravitational constant, m is the mass of the rubble object. If we express the mass of the spherical body M as

$$M = \frac{4}{3}\pi\rho R^3 , \qquad (2.4)$$

where we assume constant density  $\rho$  of the asteroid, and the velocity v of the small rubble object on the surface of the spherical body as

$$v = \omega_{crit} R , \qquad (2.5)$$

we can modify equation 2.3 to calculate the limit angular speed  $\omega_{crit}$ 

$$\omega_{crit} = \sqrt{\frac{8G\pi\rho}{3}} . \tag{2.6}$$

The last equation tells us that the critical angular speed of an asteroid does not depend on its mass or size but only on its density. It must be noted that this calculation works only for rubble piles asteroids that are bound solely by gravity.

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Figure 2.2: Astrocentric cartesian coordinate system bound to the asteroid. It is a co-rotating coordinate frame. Axes z corresponds to the rotational axis, is the angular speed of the asteroids rotation. Adapted from Mikulecká (2013).



Figure 2.3: Ecliptic coordinate system. Ecliptic coordinates  $\lambda$  and  $\beta$  show how rotation pole is determined. Adapted from Mikulecká (2013).



Figure 2.4: A graph showing the distribution of some Main Belt and Near-Earth asteroids, rotational frequency depending on the diameter of the asteroid. Adapted from Bertotti & Farinella & Vokrouhlický (2003).

### **3** The Photometry of Asteroids

Light curves of asteroids are the most efficient way to derive their global physical properties, such as three-dimensional shape, rotation period, and rotation axis orientation. Information about physical properties can be also revealed by radar imaging and stellar occultation timings.

#### 3.1 Light curves of Asteroids

A light curve is the time dependency of an object's brightness or of its apparent magnitude. From the shape of the light curve, we can estimate the nature of the variability and some of the physical properties of the observed object.

Brightness is usually measured in a specific frequency band or photometric filter, and then converted to magnitude via the Pogson equation. Often we only know the magnitude relative to a suitably chosen comparison star, which apparent magnitude we assume to be constant. In such a case, the light curve will be the time dependence of the relative magnitude  $\Delta m$ 

$$\Delta m = -2.5 \log \frac{j_{\rm v}}{j_{\rm c}} , \qquad (3.7)$$

where  $j_v$  is the brightness of the observed object, and  $j_c$  is the brightness of the comparison star. If we know the apparent magnitude of the comparison star, we can measure the time dependence of the apparent magnitude m(t) of the object directly.

The time is usually given in Julian dating<sup>11</sup>. Apparent magnitude is given in magnitudes and is usually plotted so that the curve rises with the increasing brightness of the object and vice versa.

The light curve of an asteroid is uniquely determined by

- the geometry of the observation,
- the shape of the asteroid,
- and the properties of its surface.

Asteroids are observable due to the Sun's light reflecting off their surface. As the asteroid moves and spins, it reflects light differently. Various factors cause the brightness of an asteroid to change, such as

<sup>&</sup>lt;sup>11</sup>This is a dating used primarily in astronomy. Its principle lies in counting days from a chosen beginning, which was set to the noon of UT, January 1, 4713 BC of the Julian calendar. That is the date of both the Solar and Lunar cycles start. The decimal fraction represents the time elapsed from noon of that day.

- a change of the asteroid's distance to the Earth,
- a phase change due to the asteroid orbiting around the Sun,
- the rotation of the irregular-shaped asteroid.

For an asteroid with a well-known orbit, effects caused by the asteroid orbiting the Sun can be numerically calculated. It is then possible to remove these contributions from the measured light curve and end up with a light curve dependent solely on the rotation of the asteroid [e4].

The visual magnitude of an asteroid that an observer would register if the asteroid was placed

- 1 au away from them,
- 1 au from the Sun,
- and at a solar phase angle  $\alpha$  equal to zero

is called the absolute magnitude H of an asteroid. Generally, the visual magnitude m can be calculated as

$$m = H + 5 \cdot \log \Delta_{\rm S} \Delta_{\rm E} - 2.5 \cdot \log q(\alpha) , \qquad (3.8)$$

where  $\Delta_{\rm S}$  and  $\Delta_{\rm E}$  are distances from the asteroid to the Sun and the Earth respectively.  $q(\alpha)$  is a phase integral which value can be approximated by modelling.

#### 3.2 Albedos

When radiation from the Sun hits the surface of an asteroid, some of the radiation gets absorbed and some reflected. We can measure the ratio of reflected radiation to all of the incoming radiation. This ratio is called the albedo. It is measured on a scale from 0 to 1, where 0 is for a black body absorbing all incoming radiation, and 1 corresponds to a body that reflects all incident radiation.

There are several types of albedos that are not to be confused with each other.

- Single-scattering albedo regards only one particle.
- Hemispheric albedo is the ratio of radiation reflected by one area of an object to the incoming radiation.
- Normal albedo is measured when the radiation comes perpendicularly to one area, and we observe it perpendicularly as well.
- Geometric albedo is the ratio of actual brightness of an object at zero phase angle to that of a Lambertian<sup>12</sup> disk with the same cross-section.
- Bond albedo assumes the object to be of a spherical shape.

Albedos usually depend on the material on the surface. For an asteroid, the power of the reflected radiation also depends on the current orientation of the asteroid.

<sup>&</sup>lt;sup>12</sup>The Lambertian surface is an idealized flat, fully reflecting surface, that has the same apparent brightness regardless of the angle that it is observed at.

#### **3.3 Direct method**

The so-called direct method is used for computing a light curve that would be observable from the Earth at given time and position of the asteroid. It is computed from a polyhedral convex shape model and orbital and rotational parameters.

Second-order scattering is negligible for low albedos, due to this the method only needs to check which parts of the surface are visible to both the Earth and the Sun. For this to work, we need to have the shape as a polyhedron with triangles as facets.

For a facet ds that is both visible and illuminated, we can calculate its contribution dL to the total brightness of the asteroid as

$$dL = S \cdot \bar{\omega} ds , \qquad (3.9)$$

where S is the scattering law and  $\bar{\omega}$  is the albedo. Lambert's scattering law in a simple form can be written as

$$S_{\rm L} = \mu \cdot \mu_0 , \qquad (3.10)$$
  
$$\mu = \vec{E} \cdot \vec{n} ,$$
  
$$\mu_0 = \vec{E}_0 \cdot \vec{n} ,$$

where  $\vec{E}$  is the unit vector toward the observer,  $\vec{E_0}$  is the unit vector toward the Sun, and  $\vec{n}$  is the surface unit normal.

#### **3.4 Inversion problem**

In the inversion problem, we have an observed light curve, and we're looking for parameters such that the light curve determined by the direct method from these parameters is similar to the observed one as closely as possible. If the rotation parameters and scattering properties are known, Kaasalainen & Torppa (2001) showed that the convex shape of an asteroid can be deduced from its light curves accurately.

We can write the convex inverse problem as

$$\dot{L} = A \cdot \vec{g} , \qquad (3.11)$$

where  $\vec{L}$  is the vector of the observed brightnesses,  $\vec{g}$  contains the areas of the facets of the polyhedron. A is a matrice obtained as

\_

$$A_{ij} = S_j \left( \mu^{(ij)}, \mu_0^{(ij)} \right) \bar{\omega}_j , \qquad (3.12)$$

where  $S_j$  and  $\bar{\omega}_j$  are the scattering law and albedo at the facet j, and

$$\boldsymbol{\mu}^{(ij)} = \vec{E}_i \cdot \vec{n}_j \,, \tag{3.13}$$

$$\mu_0^{(ij)} = \vec{E_{0i}} \cdot \vec{n}_j \,. \tag{3.14}$$

The typical solution of the equation 3.11 is by minimizing the square norm

$$\chi^2 = \left\| \left| \vec{L} - A \cdot \vec{g} \right| \right\|^2 \,. \tag{3.15}$$

As Kaasalainen & Torppa (2001) point out, the observed brightnesses at large solar phase angles are usually smaller than near opposition. Thus, it is beneficial to replace the standard square norm with a renormalized

$$\chi_{\rm ren}^2 = \sum_{i} \left\| \frac{L^{(i)} - A^{(i)} \cdot \vec{g}}{L^{(i)}} \right\|^2 , \qquad (3.16)$$

where  $L^{(i)}$  is the mean brightness of the *i*-th light curve. This format normalizes each light curve, causing it to oscillate around unity. Due to that, each observing geometry obtains equal weights.

More in-depth description of the inversion problem can be found in Kaasalainen (2001) and Kaasalainen & Torppa (2001).

# 4 Investigating of the inversion method

For this thesis, I wrote about two dozen small Python programs. Some of these programs process observations or modify the format of input data, convert output files into formats digestible for other programs, create batch files used to automate the entire process, generate modified input files, graphically display results, compare data, and others.

As part of this thesis, there was a need to systematically handle the files, and computing capacity used. Over 165,800 files were created for each object, with light curves, different numbers of sessions, output files, object shape files, orbital element files, and more. The CPU problem will be described in the following chapters.

The photometric data used as input light curves for this thesis are accessible in the DAMIT database [e5], which is operated by The Astronomical Institute of Charles University in Prague, Czech Republic. The database collects 3D asteroid models obtained using the inversion method. It currently contains more than 6000 models for over 3000 asteroids.

#### 4.1 The direct method in practice

A direct method calculation can be done by using *lcgenerator* script [e5]. A polyhedral convex shape model with triangular surface facets is required, as well as the ecliptic pole coordinates  $\lambda$ ,  $\beta$  given in degrees, the rotational period *P* given in hours, the initial "zero time"  $t_0$  in Julian dating and the initial rotation angle of the asteroid  $\varphi_0$ .

An input file containing light curve data and the corresponding geometry is also needed. The first line gives the total number of observations (from here onward, they are going to be referred to as *sessions*). Each session starts with the number of points and a digit for a relative (0) or calibrated (1) light curve. Then on each line, the individual light curve follows with its epoch in JD, the brightness in intensity units and cartesian coordinates x, y, z of the asteroid co-rotating coordinate frame of the Sun and of the Earth in au. For a slowly moving main-belt asteroid, the coordinate vectors can be approximated to be constant for a single-night light curve.

The result from the direct method is a text file containing calculated brightness in intensity units. The list of brightness values is written in the same order as in the input light curve file, excluding notes about the number of points etc. If the individual light curves are relative, they are normalized.

#### 4.2 The light curve inversion method in practice

The inversion method calculation has a lot more practical usage than the direct method, as we usually first have the orbital parameters and light curves before we know the shape of the observed object. Nevertheless, the direct method is put to use even during the inversion method script.

To do a calculation using the inversion method, a trinity of scripts is usually used: *convexinv*<sup>13</sup>, *minkowski\_stdinout*, and *standardtri\_stdinout*.

• *convexinv* computes a model that returns the best fit to the input lightcurves, incorporating shape, spin and scattering. Unlike equation 3.16, relative chi-square defined as follows is used

$$\chi_{\rm rel}^2 = \sum_{i} \left\| \left| \frac{L_{\rm obs}^{(i)}}{\bar{L}_{\rm obs}^{(i)}} - \frac{L^{(i)}}{\bar{L}^{(i)}} \right\|^2 , \qquad (4.17)$$

where  $L_{obs}^{(i)}$  and  $L^{(i)}$  are observed and modelled light curves that are renormalized through their average brightnesses  $\bar{L}_{obs}^{(i)}$  and  $\bar{L}^{(i)}$ . The shape representation this procedure obtains is the Gaussian image of a convex polyhedron, which is the areas of the facets with their outward normals. If the ecliptic pole coordinates are set as free parameters, the direct method calculates a light curve for the resulting shape and different options of ecliptic poles. This light curve is compared to the input light curves, and this way ecliptic pole coordinates are found.

- The Minkowski procedure in the script *minkowski\_stdinout* calculates the vertices of facets calculated by the *convexinv* script. Because it is solved iteratively, this procedure takes a little bit of time.
- For some purposes, it is useful to convert the polyhedron from *minkowski\_stdinout* to a polyhedron that has all facets triangular. That is exactly what the script *stan-dardtri\_stdinout* does it creates a polyhedron with triangular facets and puts it to the standard shape output.

The input file for the inversion method  $-input\_convexinv.txt - is$  the same as for the direct method. The asteroid's initial ecliptic pole coordinates  $\lambda$ ,  $\beta$ , the rotation period P, zero time  $t_0$ , initial rotation angle  $\varphi_0$  and other parameters are also required. The ecliptic pole coordinates  $\lambda$ ,  $\beta$  and the rotation period P can be both set as a fixed or free parameter. A file containing light curves clustered in individual sessions -lc.txt – is also required. The terminology of these two files is important, as it will be used from here onward.

The standard shape output - *shape.TRI* - can be converted to wavefront .obj format, often used for 3D object modelling. After converting the output, I have done a visualization of the asteroid's shape using one of my scripts. Such visualization can be seen in figure 4.5.

<sup>&</sup>lt;sup>13</sup>The program was written in Fortran by Mikko Kaasalainen and converted to C language by Josef Ďurech. It is accesible with documentation at [e5].



Figure 4.5: A 3D model of asteroid (21) Lutetia obtained through the light curve inversion method. The ecliptic pole was fixed at values  $\lambda = 52^{\circ}$  and  $\beta = -6^{\circ}$ . The left and middle figures are views with  $\beta = 0^{\circ}$  for both of them and  $\lambda = 0^{\circ}$  and  $\lambda = 90^{\circ}$ , in this order. The figure on the right is a top view ( $\lambda = 0^{\circ}$ ,  $\beta = 90^{\circ}$ ) of the asteroid.

#### 4.3 Search for the ecliptic pole

The objective of this thesis is to specify the requirements that should be met by the observed data so that the results of the inversion method are as unambiguous as possible. The inverse method has ambiguity problems when the input data is of poor quality or insufficient quantity. However, the numerical parameterization of these properties is not straightforward.

I utilized the process of creating light curves for the selected ecliptic pole via the direct method. For each studied object, I have created a batch file in which the *convexinv* script is repeatedly called with progressively changing ecliptic pole parameters. The execution of this entire file will be called a *computation*. The ecliptic pole is changed with a given interval to cover the entire range of coordinates. Once the calculation for one pair of fixed parameters finishes, one of my scripts compares the input light curves with the one returned by the *convexinv* script using a simple root main square method. This one RMS value is saved together with the ecliptic pole coordinates. A new calculation for the next pair of fixed parameters ensues. The cycle repeats until the full range is computed. The result is a file with ecliptic pole coordinates and their respective RMS.

This file becomes an input file for another one of my scripts. A heatmap is plotted with  $\lambda$  and  $\beta$  coordinates as axes and the RMS value as a colour magnitude of the graph. One computation results in one heatmap.

During the practical part of this thesis I encountered several problems with the heatmap approach:

• Since one *convexinv* calculation occupies one CPU core, the entire computation of one assignment takes a very long time, with the CPU running at only 13 per cent. For this reason, I divided a single computation into six batch files running in parallel, each computing on its own core. In this way, one full computation took from 4 to 14 hours. There were 9 to 14 of these computations for each object. With decreasing amount of sessions, the computation time also lowered. The total time for all the computations was around 800 CPU hours.

- For some of the rotational poles, the script did not converge, the reason for it being a ingular matrix encounter. The RMS values for such coordinates were calculated as the highest RMS multiplied by a chosen factor, usually by 1.1 or less. For the objects shown in this thesis, eventually, no graph includes a point created by this approach.
- For targets that move close to the ecliptic plane<sup>14</sup>, we must take into account the ambiguity theorem that was worded and proven by Kaasalainen & Lamberg (2006). This theorem introduces the ambiguity of the spin direction, where we are unable to distinguish between two different orientations in space differing only by  $\pi$  in the  $\lambda$  coordinate of the ecliptic pole and the z-axis of the astrocentric co-rotating frame inverted. Therefore a vertical mirror-image shape with a rotation direction changed has the same viewing and illumination conditions and yields the same observations as the original body.

While plotting the heatmaps, I also noticed a vertical mirroring of my ecliptic poles. For the objects I used in this thesis, their orbital planes were close to the ecliptic plane and the acquired ecliptic poles differed by  $\pi$ . Thus I presume the ambiguity theorem is the cause of this mirroring.

After acquiring a heatmap, I wanted to highlight the position of the minimum. Because of the nature of the data, scipy.optimize methods of minimalization were out of the question. Therefore, I plotted the heatmap only for the directions of the ecliptic pole whose RMS value was close to the global minimal RMS value  $RMS_{min}$ . Two thresholds of the vicinity were used; 20 per cent and 10 per cent of the difference between the RMS extremes. Leaving other data boxes at zero would make them lower than the actual minimum. For this reason,



Figure 4.6: A graph showing a heatmap of object (9) Metis. The ecliptic pole coordinates changed with a step of 5°. The x axis shows  $\lambda$  in degrees. The left figure shows the complete heatmap. The figure in the middle and on the left show only configurations of the ecliptic pole whose RMS differed from RMS<sub>min</sub> by 20 per cent and 10 per cent of RMS<sub>max</sub> – RMS<sub>min</sub> at maximum. The rest of the values were left at the arithmetric mean of RMS<sub>max</sub> and RMS<sub>min</sub>, mainly for visibility purposes. The minimal RMS = 17.78 mmag is located in the box with coordinates  $\lambda = 180^\circ$ ,  $\beta = 20^\circ$ . The secundary minimum RMS = 18.07 mmag is located in the box with coordinates  $\lambda = 360^\circ$ ,  $\beta = 5^\circ$ .

<sup>&</sup>lt;sup>14</sup>The ability to differentiate is influenced by the amount of photometric data from the time when the asteroid moves farthest from the ecliptic plane. In practice, this theorem affects asteroids with an inclination up to 20° from the ecliptic plane (Mikulecká, 2013).



Figure 4.7: A heatmap of object (21) Lutetia. The figure has the same settings as figure 4.6. The minimal value of RMS = 14.65 mmag is located in the box with coordinates  $\lambda = 55^{\circ}$ ,  $\beta = -10^{\circ}$ . The secundary minimum RMS = 14.82 mmag is located in the box with coordinates  $\lambda = 235^{\circ}$ ,  $\beta = -5^{\circ}$ .



Figure 4.8: A heatmap of object (29) Amphitrite. The figure has the same settings as figure 4.6. The minimal RMS = 12.65 mmag is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -20^{\circ}$ . The secundary minimum RMS = 12.78 mmag is located in the box with coordinates  $\lambda = 320^{\circ}$ ,  $\beta = -30^{\circ}$ .



Figure 4.9: A heatmap of object (39) Laetitia. The figure has the same settings as figure 4.6. The minimal RMS = 14.74 mmag is located in the box with coordinates  $\lambda = 320^{\circ}$ ,  $\beta = 30^{\circ}$ . The secundary minimum RMS = 16.34 mmag is located in the box with coordinates  $\lambda = 130^{\circ}$ ,  $\beta = 25^{\circ}$ .

I assigned the rest of the boxes the value of the arithmetic mean. Such visualization for object (9) Metis can be seen in figure 4.6.

The heatmap of (21) Lutetia is shown in figure 4.7, in figure 4.8 a heatmap of (29) Amphitrite is plotted, figure 4.9 shows the heatmap of (39) Laetitia. All four objects show signs of the ambiguity theorem, with their minimums mirrored. Differences between primary and secondary minimums range from 0.13 mmag to 1.6 mmag. The areas of minimal RMS are fairly small, and their ecliptic poles are located quite well. A comparison between the values from databases and the values from figures 4.6 to 4.9 is illustrated in table 4.2. In figure 4.9 we can observe the largest range of RMS values, where the minimum is 14.74 mmag, and the maximum is 83.49 mmag.

Table 4.2: A table with the ecliptic pole coordinates. Database coordinates  $\lambda$ ,  $\beta$  for (21) Lutetia are from Carry et al. (2010), values for the rest of the objects are from Hanuš et al. (2013). The primary minimum has coordinates  $\lambda_{hm_{-1}}$ ,  $\beta_{hm_{-1}}$ , the secondary minimum has coordinates  $\lambda_{hm_{-2}}$ ,  $\beta_{hm_{-2}}$ . For the primary minimum of (9) Metis, a heatmap with step of 1° was used (figure 4.10). For the primary minimum of (29) Amphitrite, a heatmap with step of 1° was used (figure 4.12).

	λ [°]	β [°]	$\lambda_{\text{hm}\_1}$ [°]	$eta_{ ext{hm}_{-1}}$ [°]	λ <sub>hm_2</sub> [°]	β <sub>hm_2</sub> [°]
(9) Metis	182	19	182	20	360	5
(21) Lutetia	52	-6	55	-10	235	-5
(29) Amphitrite	322	-28	137	-20	320	-30
(39) Laetitia	323	32	320	30	130	25

Upon looking closely at the heatmaps, one can notice that the minimal RMS doesn't go under the value of 13 mmag. While this is a very good result, I wanted to know how far I can push the limits. For the correct configuration of the ecliptic pole, a value of minimal RMS close to zero would be expected, as well as a small area of boxes within the chosen thresholds. The signal noise in the observed light curves might be the cause of not getting the desired results.

I smoothed the light curves by applying the moving average filter using convolve function from the NumPy module. This function calculates a discrete convolution as

$$(a*v)_n = \sum_{m=-\infty}^{\infty} a_m \cdot v_{n-m} , \qquad (4.18)$$

where a and v are two one-dimensional sequences (in my case, a is a sequence from the original light curve, and v is a sequence of thirteen numbers with a value of 1/13). The function filters the whole input and returns an array of the same length with noise smoothed out. Each session was modified separately. It was necessary to resolve the problematic edges, thus I added a requirement to keep the original value  $a_n$  if the new value  $(a * v)_n$  returned by the filter diviated from the original value by more than 25 per cent of the difference between extremes.

I computed the heatmap for (9) Metis with coordinates of ecliptic pole in limited intervals  $\lambda \in \langle 162, 198 \rangle^{\circ}$ ,  $\beta \in \langle -8, 53 \rangle^{\circ}$ , which corresponds to the 20 per cent threshold from figure 4.6. The coordinates changed with a step of 1°. There were two computations,



Figure 4.10: A graph showing a heatmap of object (9) Metis in the interval of coordinates highlighted in figure 4.6 for the 20 per cent threshold. The ecliptic pole coordinates changes with a step of 1°. The left figure shows the complete heatmap. The figure on the right shows only boxes with deviation smaller than 20 per cent of difference between the RMS extremes. Note that this 20 per cent threshold would be equal to a 4 per cent threshold in figure 4.6. The rest of the values were left at the arithmetric mean of RMS<sub>max</sub> and RMS<sub>min</sub>. The minimum sits at coordinates  $\lambda = 182^{\circ}$ ,  $\beta = 20^{\circ}$ , and has a value of RMS = 17.71 mmag.



Figure 4.11: A graph showing a heatmap of object (9) Metis with the same settings as in figure 4.10. Modified light curves were used as an input. The minimum sits at coordinates  $\lambda = 180^{\circ}$ ,  $\beta = 19^{\circ}$ , and a value of RMS = 17.27 mmag.



Figure 4.12: A graph showing a heatmap of object (29) Amphitrite with coordinates in limited intervals  $\lambda \in \langle 120, 150 \rangle^{\circ}$ ,  $\beta \in \langle -53, -1 \rangle^{\circ}$ , which roughly corresponds to the 20 per cent threshold of primary minimum from figure 4.8. The figure has the same settings as figure 4.10. The minimum sits at coordinates  $\lambda = 137^{\circ}$ ,  $\beta = -20^{\circ}$ , and has a value of RMS = 12.61 mmag.



Figure 4.13: A graph showing a heatmap of object (29) Amphitrite with the same settings as in figure 4.12. Modified light curves were used as an input. The minimum sits at coordinates  $\lambda = 137^{\circ}$ ,  $\beta = -21^{\circ}$ , and has a value of RMS = 11.45 mmag.

Table 4.3: A table with the minimal RMS values for different computation. In the first column, there are results of computation with a step of  $5^{\circ}$ . Second column shows results of computation with a step of  $1^{\circ}$  and non-modified input data (figures 4.10 and 4.12). Second column shows results of computation with a step of  $1^{\circ}$  and non-modified input data (figures 4.11 and 4.13).

	RMS <sub>5°</sub> [mmag]	RMS <sub>1°, non-modif</sub> [mmag]	RMS <sub>1°, modif</sub> [mmag]
(9) Metis	17.78	17.71	17.27
(29) Amphitrite	12.65	12.61	11.45



Figure 4.14: A graph showing three representative light curves (sessions) of object (9) Metis. In the top row, a comparison of the input data is plotted. The input non-modified light curves are shown as yellow dots, the input modified data is plotted as smaller blue dots. A comparison of the output data is plotted in the bottom row. The data obtained through non-modified input for  $\lambda = 182^{\circ}$ ,  $\beta = 20^{\circ}$  is in red, while data from modified input light curves for  $\lambda = 180^{\circ}$ ,  $\beta = 19^{\circ}$  is in blue. The light curves are plotted in relative magnitudes with  $j_0 = 2.54$  units and  $m_0 = 0$  mag.



Figure 4.15: A 3D models of asteroid (9) Metis obtained through the light curve inversion method. The left model is computed with non-modified input data for  $\lambda = 182^{\circ}$ ,  $\beta = 20^{\circ}$ , for the model on the right the input light curves were smoothened and the ecliptic pole was fixed at  $\lambda = 180^{\circ}$ ,  $\beta = 19^{\circ}$ .



Figure 4.16: A graph showing three representative light curves (sessions) of object (29) Amphitrite. In the top row, a comparison of the input data is plotted. The input non-modified light curves are shown as yellow dots, the input modified data is plotted as smaller blue dots. A comparison of the output data is plotted in the bottom row. The data obtained through non-modified input for  $\lambda = 137^{\circ}$ ,  $\beta = -20^{\circ}$  is in red, while data from modified input light curves for  $\lambda = 137^{\circ}$ ,  $\beta = -21^{\circ}$  is in blue. The light curves are plotted in relative magnitudes with  $j_0 = 2.54$  units and  $m_0 = 0$  mag.



Figure 4.17: A 3D models of asteroid (29) Amphitrite obtained through the light curve inversion method. The left model is computed with non-modified input data for  $\lambda = 137^{\circ}$ ,  $\beta = -20^{\circ}$ , for the model on the right the input light curves were smoothened and the ecliptic pole was fixed at  $\lambda = 137^{\circ}$ ,  $\beta = -21^{\circ}$ .

one for the original and one for the modified data. The results are depicted in figures 4.10 and 4.11. I have repeated the same process for (29) Amphitrite in the interval of coordinates  $\lambda \in \langle 120, 150 \rangle^{\circ}$ ,  $\beta \in \langle -53, -1 \rangle^{\circ}$ , both for the original and modified data. The coordinates also changed with a step of 1°. These results are depicted in figures 4.12 and 4.13.

For object (9) Metis, apart from a slight shift in the position of the minimum, no significant difference is visible at first glance. By smoothening the input data, I was able to improve the minimal RMS value only by half of a milimagnitude. For object (29) Amphitrite, there is also a small shift in the position of the minimum, but the minimal RMS value improved by more than a milimagnitude.

If we look at figures 4.14 and 4.16, we can see how the smoothed data (blue dots) look compared with the original non-modified data (plotted as yellow dots). The second rows of the figures show how little the smoothening changed the output. We can see that even with the non-modified data, the output light curves (in red) match the input well.

Figures 4.15 and 4.17 show a comparison of the shapes created by non-modified and modified input data. The newly located minimum becomes the input ecliptic pole for each model. In figure 4.15 for object (9) Metis, a minor change of the shape and the facets is noticeable. For the object (29) Amphitrite in figure 4.17, one must look very closely to find a difference. Changes in the sizes of the facets can be discerned at the model's three o'clock and eleven o'clock.

In figures 4.14 and 4.16, there is no noticeably better output of modified data for any of the two objects. Interestingly, the smoothening affected the heatmap of (9) Metis less than the heatmap of (29) Amphitrite (figures 4.14 and 4.16), while models for (9) Metis are more different from each other than models of (29) Amphitrite (figures 4.15 and 4.17).

In table 4.3, there are minimal RMS values for different input data and step sizes. While reducing the step size from  $5^{\circ}$  to  $1^{\circ}$  is a logical decision, the RMS value only improved in order of hundredths of milimagnitude. Such improvement is negligible whereas computing the entire heatmap with a step size of  $1^{\circ}$  would take substantially more time than using a  $5^{\circ}$  step size. The usage of modified data to some extent helps to determine the minimum better, whereas, during the process of modification, we lose some of the physical aspects of the solution.

### 5 Reduction of the input data

In the previous chapter, I concentrated solely on comparing the results of the inversion method with the input data for different orientations of the rotational axis. I have always worked with the full amount of sessions from the database. In this chapter, I study the effects of decreasing the amount of input data.

I first observed the change in how the heatmap looks based on how many sessions were used to compute it. For each new input file, I first selected the number of sessions I wanted to remove from the original input file, and a random generator picked the sessions to remove. To create new reduced input files, the original unmodified file was used, so that what sessions remained were not interdependent on the previously modified file. This way, I created several new input files for each object with a progressively decreasing amount of sessions.



Figure 5.18: A graph showing what effects the decreasing amount of input sessions has on the final heatmap graph. Heatmaps were computed for (9) Metis across the whole interval of coordinates. The colourbar shows RMS in milimagnitudes.



Figure 5.19: A graph with heatmaps of (21) Lutetia with the same settings as figure 5.18.



Figure 5.20: Heatmaps of (29) Amphitrite with the same settings as figure 5.18.



Figure 5.21: A graph with heatmaps of (39) Laetetia with the same settings as figure 5.18.



Figure 5.22: A heatmap of object (39) Laetitia. The figure has the same settings as figure 4.6. The minimal RMS = 9.60 mmag is located in the box with coordinates  $\lambda = 325^{\circ}$ ,  $\beta = 35^{\circ}$ . The secundary minimum RMS = 10.40 mmag is located in the box with coordinates  $\lambda = 130^{\circ}$ ,  $\beta = 30^{\circ}$ .

In figure 5.18, we can observe how the area of the minimal RMS increases in size for asteroid (9) Metis. While in the heatmaps computed with down to 22 input sessions we can still determine where the two mirroring minimums are, heatmaps from 18 and fewer input sessions don't have a clear minimal RMS area. Whether or not the minimum can be located is determined by having two areas in the 20 per cent threshold from  $RMS_{min}$  at maximum. For figure 5.19, we can locate the minimum of (21) Lutetia down to 26 input sessions. In figure 5.20, the minimal RMS of (29) Apmhitrite can be located even with the heatmap with 28 input sessions. For these three asteroids, the minimum can be located
with only 58 per cent for (9) Metis, 52 per cent for (21) Lutetia and 42 per cent for (29) Amphitrite of their input sessions.

As for figure 5.21, the minimal RMS of object (39) Laetitia was localizable even with only 18 per cent of its input sessions. Even for 12 input sessions, only two mirrored minimums passed the 20 per cent threshold, as is shown in figure 5.22. The position of the minimum shifted slightly, more in the  $\beta$  coordinate than the  $\lambda$  coordinate, but it is still close to the values from table 4.2.

#### 5.1 Spatial distribution

The distribution of different positions of the examined asteroid plays an important role. After removing sessions, if the only left sessions cover a low portion of the asteroid's orbit, the search for the ecliptic pole will be hampered. In this section, I will be examining the effects the geometry of the observation has on the heatmaps.

In the data from the DAMIT database [e5], the position of the asteroid is noted for the astrocentric coordinate system, with a vector pointing towards the Earth ( $x_E$ ,  $y_E$ ,  $z_E$ ) and a vector pointing towards the Sun ( $x_S$ ,  $y_S$ ,  $z_S$ ). Instead of converting these values, I utilized the Horizons System API [e6] to acquire the cartesian vectors in a specific reference frame. The input time stamps are the beginning of each session. For my thesis, the reference frame used was the 'Ecliptic of J2000.0', and the centre body of this frame was the Solar System barycenter.

The output of such ephemeris generation is a table with the same amount of rows as input sessions and columns with values of x, y and z vectors in astronomical units. If plotted as it is, one ends up with an ellipse in a 3D graph. For my purposes, I wanted to plot the ellipse in a 2D graph by rotating the orbit in such a way, that the z coordinate would be constant, and therefore I could omit it. A simple projection to the x - y (ecliptic) plane would deform the shape of the orbit, so for this reason, I performed the change of the basis.

Let us consider two coordinate systems for vectors in vector space. If X and X' are the coordinates of the same vector in two different coordinate systems B and B', we can use the change-of-basis matrix P to write

$$X = P \cdot X'$$

for the vector transition from base B to B'.

By taking three points – let's call them A, B, C – from the asteroid's orbit, we can get the cross product  $\vec{n}$  of vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  as  $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{n}$ . Normalized vector  $\overrightarrow{AB}$  becomes the basis vector x', and normalized vector  $\vec{n}$  becomes the basis vector z'. The cross product of  $x' \times z'$  becomes the last basis vector y'. The change-of-basis matrix P is created as (x', y', z'), thus it has in its columns the coordinates of the vectors of base B' expressed in base B. A dot product of the matrix P and the coordinates from the Horizons System yields the desired two-dimensional coordinates of the asteroid's orbit.

The surface of the area enclosed by the elliptical orbit is required for further examination. For this purpose, a direct linear least squares fitting of an ellipse was done with a Python code accessible at [e7]. I labelled the ratio between the covered area and the area of the ellipse as Coverage and marked it with the letter *C*. For the coverage *C* of an asteroid



Figure 5.23: A graph showing the spatial distribution for different numbers of input sessions for object (9) Metis. The grey ellipse is a fit done by direct linear least squares fitting. Blue dots are the positions of the asteroid acquired from the Horizons System. The coverage C is calculated as a ratio between the red area and the full area of the ellipse. The subplot for 2 input sessions (corresponding to the one in figure 5.18) is missing because with two points it is not possible to create a basis and to fit an ellipse.

to be close to 1, observations had to be made in various geometries throughout the entire orbital range. As the number of input sessions decreases randomly, the orbit coverage does not have to drop as well. Good coverage for specific sessions can explain their better results on the heatmap than for the same number of other selected sessions.

Figure 5.23 shows the spatial distribution for object (9) Metis. The sessions used for each of the subplots correspond to the sessions used for computing the heatmaps in figure 5.18. The same goes for object (21) Lutetia and figures 5.24 and 5.19. For object (29) Amphitrite, spatial distribution in figure 5.25 matches the sessions that the heatmaps in figure 5.20 were made with. As for (39 Laetitia in figure 5.20, its corresponding spatial distribution is in figure 5.25. The spatial distribution for the lowest number of sessions is missing in figures 5.23, 5.24 and 5.26 for reasons such as not being able to create a base with only two points and unsuccessful fitting of the ellipse.

Table 5.4 lists the spatial distribution coverages for the border number of input sessions for each of the examined objects. For the least number of input sessions for which the minimum is still localizable, the average coverage was  $C_{LL} = 0.82 \pm 0.13$ . For the greatest number of input sessions for which the minimum was not localizable, the average coverage was  $C_{FNL} = 0.75 \pm 0.07$ .



Figure 5.24: A graph with the spatial distribution for object (21) Lutetia. The figure has the same setting as 5.23 and matches to figure 5.19.



Figure 5.25: A graph with the spatial distribution for object (29) Amphitrite. The figure has the same setting as 5.23 and matches to figure 5.20.



Figure 5.26: A graph with the spatial distribution for object (39) Laetitia. The figure has the same setting as 5.23 and matches to figure 5.21. The subplot for 4 input sessions is missing because the ellipse fitting was not successful in this case.

Table 5.4: A table with the spatial distribution coverages *C* for a different number of input sessions. The last localizable (LL) and first non-localizable (FNL) settings were determined by figures 5.18, 5.19, 5.20 and 5.21. The coverages  $C_{LL}$  and  $C_{FNL}$  for the corresponding number of sessions are following figures 5.23, 5.24, 5.25 and 5.26.

	Last localizable	$C_{\rm LL}$	First non-localizable	$C_{\rm FNL}$
(9) Metis	22 sessions	0.92	18 sessions	0.81
(21) Lutetia	26 sessions	0.79	20 sessions	0.68
(29) Amphitrite	28 sessions	0.92	20 sessions	0.77
(39) Laetitia	12 sessions	0.65	4 sessions	_



Figure 5.27: The spatial distribution for 28 input session of (29) Amphitrite and the heatmap obtained through the light curve inversion method using these input sessions. The minimal  $RMS_{min} = 11.24$  mmag is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -20^{\circ}$ .



Figure 5.28: The spatial distribution and the heatmap for 28 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.27 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}, \beta = -20^{\circ}$ .

#### 5.1.1 Limits of spatial distribution for (29) Amphitrite

To better determine the limit of the coverage for which the ecliptic pole minimum is still localizable, I further investigated (29) Amphitrite. For the complete set of input sessions, this object has the minimum located with the lowest  $RMS_{min}$  value. This asteroid also did not encounter any problem with plotting the spatial distribution graph for less than 6 sessions.

I focused on the border settings of (29) Amphitrite, which were 28 and 20 input sessions. I created five more of each of these settings, which means six different initial settings with 28 input sessions and six different initial settings with 20 input sessions in total. I computed heatmaps for these settings and compared the coverages for each spatial distribution. The graphs can be seen in figures 5.27 to 5.30 and 6.33 to 6.38. For all of these figures, the RMS value of the boxes that don't meet the 20 per cent threshold was set at 20 mmag. The results are also listed in table 5.5.



Figure 5.29: The spatial distribution and the heatmap for 20 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.33 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}, \beta = -20^{\circ}$ .



Figure 5.30: The spatial distribution and the heatmap for 20 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 10.05 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}, \beta = -20^{\circ}$ .

It's apparent from the graphs that just having diverse positions of the observed object doesn't guarantee a good result. Figures 5.30 and 5.29 show it nicely – while the first has a higher value of coverage *C*, the minimum is not determined well and the RMS<sub>min</sub> value is actually higher than the one for figure 5.29.

For table 5.5, I determined for each set the proportion of how many sessions consist of less than 16 points. This proportion is listed in the column Low-point. After removing these sessions with low numbers of points I recalculated the coverage  $C_{[sel]}$ . Assuming that removing the low-point sessions doesn't affect the localizability, for 15 input sessions, the coverage  $C \ge 0.76$  seems to be enough. For 16 input sessions, the limit lies somewhere between C = 0.56 and C = 0.83. For 20 input sessions, the coverage C = 0.89 or more is sufficient.

With another attempt to find the limit for coverage, I computed more heatmaps for different sets of input sessions. Only sessions with more than 15 points were selected. The resulting graphs are listed in the Appendix with numbers from 6.39 to 6.52. Table 5.6 compares the coverages, numbers of input sessions and localizability of the minimum.

In figure 5.31, I plotted the data from table 5.6. Settings with localized minimums are plotted in green, and settings with non-localized minimums are in red. I fitted the localized

Table 5.5: A table with the spatial distribution coverages *C* for different sets of input sessions. The amount of input sessions is noted in column Obs.  $C_{\text{full}}$  is the coverage of the full setting. The RMS<sub>min</sub> is as usual listed in milimagnitudes. Localizability was determined by having two areas in the 20 per cent threshold from RMS<sub>min</sub> at maximum. The column named Low-point lists the percentage of sessions with 15 or fewer points in them, whereas  $C_{\text{sel}}$  is the coverage if we select only the other sessions (with at least 16 points). Obs<sub>sel</sub> states the remaining number of input sessions, and the table is ordered by this column. The values follow the figures listed in the last column.

Obs	$C_{\text{full}}$	RMS <sub>min</sub>	Localizable	Low-point	Obs <sub>sel</sub>	$C_{\rm sel}$	Figures
20	0.64	13.70	No	0.50	10	0.49	6.36
20	0.77	11.11	No	0.35	13	0.59	5.20, 5.25
20	0.87	11.33	No	0.35	13	0.83	5.29
20	0.77	10.05	Yes	0.25	15	0.76	5.30
20	0.89	11.60	Yes	0.25	15	0.77	6.37
20	0.58	12.39	No	0.20	16	0.56	6.38
28	0.92	12.80	Yes	0.43	16	0.83	5.20, 5.25
28	0.13	9.79	No	0.36	18	0.07	6.34
28	0.46	11.27	No	0.36	18	0.37	5.28
28	0.91	10.91	No	0.32	19	0.81	6.33
28	0.90	11.24	Yes	0.29	20	0.89	5.27
28	0.94	13.72	Yes	0.29	20	0.92	6.35

Table 5.6: A table with the spatial distribution coverages C for different sets of input sessions. All input sessions have at least 16 points. Localizability was determined the usual way. The values follow the figures listed in the last column.

Sessions	C	RMS <sub>min</sub> [mmag]	Localizable	Figures
10	0.88	9.41	No	6.39
10	0.90	10.58	No	6.40
15	0.71	10.85	No	6.41
15	0.75	8.19	No	6.42
15	0.76	10.02	Yes	5.30, 6.43
20	0.77	11.59	No	6.44
20	0.82	11.68	No	6.45
20	0.89	11.12	Yes	5.27, 6.46
25	0.66	11.50	No	6.47
25	0.82	11.26	Yes	6.48
25	0.91	12.13	Yes	6.49
30	0.78	9.78	Yes	6.50
30	0.81	9.63	Yes	6.51
30	0.88	11.09	Yes	6.52



Figure 5.31: A graph plotting the data from table 5.6. Fitting was done only for localized data. The blue dashed line is the first-order polynomial  $p(x) = 0.0015 + 0.7982 \cdot x$ . The pink dashed line is the second-order polynomial  $p(x) = -0.0015 + 0.0716 \cdot x + 0.0369 \cdot x^2$ . The third-order polynomial  $p(x) = 0.0002 - 0.0142 \cdot x + 0.3512 \cdot x^2 - 1.9433 \cdot x^3$  is in orange.

data with a polynomial using the polyfit function from the NumPy module. This function fits the data with a polynomial  $p(x) = p_0 \cdot x^0 + p_1 \cdot x^1 + ... + p_k \cdot x^k$ , where  $p_i$  are the coefficients of the polynomial and *k* is the order of the polynomial, and minimizes the squared error *E* as

$$E = \sum_{j=0}^{k} \left| p(x_j) - y_j \right|^2,$$
(5.19)

where  $y_j$  are the y values of the data fitted. The blue dashed line marks a first-order polynomial, the pink dashed line is a second-order polynomial, and the orange dashed line is a third-order polynomial. Their polynomial coefficients are noted in the description of figure 5.31. Neither of the fits takes account of the 10 input sessions because there was no heatmap with only 2 minimums for no low-point sessions for (29) Amphitrite.

We can see that for such a low amount of data points, the safest option is the linear fit. For the right side of fit, a slow and steady descent or an asymptotic closing to some constant limit value could be expected. Assuming that the localizability depends only on the spatial distribution and the number of input sessions (which is unlikely), more data points would be necessary to find a good fit for such dependence. Another approach could be some Machine Learning classification to determine the boundary between localized and non-localized data.

#### 5.2 Time Distribution

In this section, I examine how well are different portions of the asteroid's rotational period covered by observations. For an object with periodically changing brightness and constant period, for long-term observation, it is sometimes beneficial to use the phase function  $\vartheta(t)$ 

$$\vartheta(t) = \frac{t - M_0}{P} , \qquad (5.20)$$

where t is the time in Julian dating, P is the rotational period and  $M_0$  is a chosen moment from which we count one rotation of the object. At each time, an epoch E(t) and a phase  $\varphi(t)$  can be calculated

$$E(t) = floor[\vartheta(t)], \qquad (5.21)$$

$$\boldsymbol{\varphi}(t) = frac[\vartheta(t)], \qquad (5.22)$$

where epoch E(t) is the counter for how many full rotations passed since  $M_0$  and the phase  $\vartheta(t)$  expresses the degree of completion of one rotation.

I plotted a histogram graph to showcase what portions of the asteroid's rotation were observed. To have good coverage of the entire asteroid rotation, we expect a uniform distribution of data in the histogram. I evaluated the histogram by the Kolmogorov–Smirnov test. It is a test of the equality of continuous one-dimensional probability distributions. We can compare one dataset with a reference probability distribution or two different datasets with each other to find out if the data has the same distribution. The quantification works by calculating the distance between the empirical distribution function of the dataset and the cumulative distribution function of the reference distribution. In my case, the reference distribution was the uniform distribution, whose cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x \le a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x \ge b \end{cases}$$
(5.23)

I implemented the Kolmogorov–Smirnov test by using stats.kstest function from the SciPy module. This function returns a test statistic value telling us how well-fitted the distributions are. If the two distributions are identical for all *x*, that is the null hypothesis; the alternative hypothesis says that the distributions are not identical. We should reject the null hypothesis if the test statistic value is too high – meaning the distances between the tested distributions are also high. With a rising data point number *n*, the critical distance decreases. I calculated my critical distances by Panik (2014) as  $1.92/\sqrt{n}$ . With a confidence level of 95%, we also reject the null hypothesis, if the p-value is less than 0.05; thus the two distributions are not identical. For the p-value higher than 0.05, the estimation of the two distributions being identical is not rejected. In figure 5.32, the time distribution for the full set of sessions for (29) Amphitrite is plotted as well as the comparison between the empiric distribution function of the time distribution and the uniform cumulative distribution function. With ht p-value = 0.02, the null hypothesis needs to be rejected and the distribution is not uniform.

Table 5.7: A table with the Kolmogorov-Smirnov test results for the time distribution of (29) Amphitrite. The table includes only set with a localizable minimum. The values follow the figures listed in the last column. The stat value needs to be lower than the stat<sub>Limit</sub> value and the p-value needs to be higher than 0.05 for us to not have to reject the hypothesis of the time distribution being uniform.

Sessions	stat	stat <sub>Limit</sub>	p-value	Figures
15	0.06	0.12	0.31	5.30, 6.43, 6.54
20	0.05	0.11	0.44	5.27, 6.46, 6.54
20	0.06	0.11	0.32	5.30, 6.54
20	0.05	0.11	0.47	6.37, 6.54
25	0.06	0.10	0.13	6.48, 6.54
25	0.04	0.09	0.37	6.49, 6.55
28	0.05	0.10	0.38	5.27, 6.55
28	0.10	0.10	0.00	6.35, 6.55
28	0.05	0.09	0.34	5.25, 6.55
30	0.07	0.09	0.01	6.50, 6.55
30	0.07	0.08	0.02	6.51, 6.55
30	0.07	0.09	0.01	6.52, 6.55
36	0.06	0.09	0.03	5.25, 6.55
44	0.05	0.08	0.05	5.25, 6.55
52	0.04	0.07	0.16	5.25, 6.56
60	0.05	0.06	0.03	5.25, 6.56
66	0.05	0.06	0.02	5.25, 5.32



Figure 5.32: The histogram showing the time distribution of (29) Amphitrite for 66 input sessions. The graph on the right is the comparison between the cumulative distribution function (CDF) of the uniform distribution (red line) and the empiric distribution function (eCDF) for the sessions used.

Table 5.8: A table with the Kolmogorov-Smirnov test results for the time distribution of (29) Amphitrite. The table includes only set with a non-localizable minimum. The values follow the figures listed in the last column. The stat value needs to be lower than the stat<sub>Limit</sub> value and the p-value needs to be higher than 0.05 for us to not have to reject the hypothesis of the time distribution being uniform.

Sessions	stat	stat <sub>Limit</sub>	p-value	Figures
4	0.07	0.27	0.94	5.25, 6.53
10	0.08	0.15	0.28	6.39, 6.53
10	0.06	0.16	0.60	6.40, 6.53
12	0.11	0.16	0.05	5.25, 6.53
15	0.09	0.13	0.07	6.41, 6.54
15	0.10	0.14	0.03	6.42, 6.53
20	0.07	0.11	0.11	6.44, 6.54
20	0.09	0.12	0.02	6.45, 6.54
20	0.09	0.14	0.10	6.36, 6.54
20	0.06	0.11	0.26	5.29, 6.54
20	0.05	0.13	0.57	6.38, 6.54
20	0.06	0.12	0.32	5.25, 6.54
25	0.04	0.10	0.64	6.47, 6.54
28	0.05	0.11	0.40	6.34, 6.55
28	0.05	0.10	0.41	5.28, 6.55
28	0.05	0.10	0.27	6.33, 6.55

In tables 5.7 and 5.8 (localized and non-localized minimum), we can see that in most cases the hypothesis of the time distribution being uniform cannot be rejected. Except for one case, the stat value (corresponding to the test statistic result of stats.kstest function) is always lower than its limit, thus labelling the fit as satisfactory.

As for the p-value, the values fluctuate wildly for both localizable and non-localizable data sets. It seems it's more important to have some observation of each portion of the rotation cycle than to have a uniform distribution of those observations. Therefore, there is no apparent correlation between the uniformity of the time distribution and the localizability of the rotational pole. For this reason, for the other 3 objects, the time distributions were not calculated.

# Conclusions

The objective of this thesis was to investigate the limitations of the inverse method (Kaasalainen, 2001) used to determine the rotational state and shape of asteroids from their photometric data and to try to find the requirements that the photometric data should meet for the method to return an unambiguous result.

When selecting objects suitable for investigation, I used photometric data and model parameters from the DAMIT database [e5]. All procedures concerning the inversion and direct methods are also accessible in this database.

It was necessary to work thoughtfully with the file system and the utilization of processor power to reduce computational time. I wrote about two dozen small Python programs for this thesis, and some of them are showcased at the end of the Appendix.

To study the accuracy of the inversion method, I calculated the shape model for different settings of the rotational axis orientation. For each setting, I compared the input light curves with the ones returned and assigned an RMS value to this coordinate setting. Such one computation results in one heatmap – during the scope of the thesis many heatmaps were computed. The area with minimal RMS value is then called the primary minimum and is labelled as the ecliptic pole of the object. The ecliptic poles were found at

- $\lambda = (182.0 \pm 0.5)^{\circ}$ ,  $\beta = (20.0 \pm 0.5)^{\circ}$ , RMS<sub>min</sub> = 17.71 mmag for (9) Metis,
- $\lambda = (55.0 \pm 2.5)^{\circ}, \beta = (-10.0 \pm 2.5)^{\circ}, \text{RMS}_{\text{min}} = 14.65 \text{ mmag for (21) Lutetia,}$
- $\lambda = (137.0 \pm 0.5)^{\circ}, \beta = (-20.0 \pm 0.5)^{\circ}, \text{RMS}_{\text{min}} = 12.61 \text{ mmag for (29)} \text{ Amphitrite,}$
- $\lambda = (320.0 \pm 2.5)^{\circ}, \beta = (30.0 \pm 2.5)^{\circ}, \text{RMS}_{\text{min}} = 14.74 \text{ mmag for (39) Laetitia.}$

The uncertainty is given by the step size of the heatmap. In table 4.2, the resulting coordinates can be compared with the ones from the database. For (29) Amphitrite, the secondary minimum matches the database values. For the other objects, most of the results are within the uncertainty.

I modified the input data of (9) Metis and (29) Amphitrite with a moving average to see if such data would improve the results. For this part, I worked with a 20 per cent threshold. For modified data, the ecliptic poles were found at

- $\lambda = (180.0 \pm 0.5)^{\circ}$ ,  $\beta = (19.0 \pm 0.5)^{\circ}$ , RMS<sub>min</sub> = 17.27 mmag for (9) Metis,
- $\lambda = (137.0 \pm 0.5)^{\circ}, \beta = (-21.0 \pm 0.5)^{\circ}, \text{RMS}_{\text{min}} = 11.45 \text{ mmag for (29) Amphitrite.}$

Figures 4.10 to 4.17 compare the resulting heatmaps, the fitting of light curves and the computed shapes for modified and non-modified input data. The position of the minimum

shifted, and as seen in table 4.3, the  $RMS_{min}$  value reduced by up to 1.2 mmag. The visual appearance of the heatmap is slightly different and there are differences in the shape model, but it's barely visible. The inversion method having some smoothing built-in appears to be a reasonable assumption. Overall, the manual preprocessing of the data is unnecessary.

I attempted to specify the dependence of the uncertainty of the result on the properties of the data. I compared the effects of the number of observations provided, their spatial distribution and the time coverage of the rotation period. For this purpose, I computed heatmaps with decreased number of sessions and observed the change in how the heatmap looked (figures 5.18 to 5.21). For each object, the limit number of input sessions differed. It ranged between 12 and 28 as the least number for which the minimum is still localizable. Randomly trimmed input data sets did not yield an explicit result for specifying a sufficient quantity of input data.

I examined the spatial distribution for each of the studied objects. I defined the coverage parameter C to denote the ratio between the area covered by observations and the full area covered by the object's orbit. For a main-belt asteroid to be modelled, at least three well-covered apparitions are necessary (?). To localize the ecliptic pole, the needed C ranged between 0.65 and 0.92, as seen in table 5.4.

I further investigated the dependence of the limit coverage on the number of input sessions for (29) Amphitrite. For coverage  $C \ge 0.92$  the minimum was always localizable, but for lower coverages, the values overlapped. For this reason, I selected only sessions that included at least 16 points and repeated the process. In figure 5.31, I plotted the results for this selection and fitted the localized data. The first-order polynomial fit has a function  $p(x) = 0.0015 + 0.7982 \cdot x$ .

In the end, I focused on examining the coverage of different portions of the asteroid's rotational period. The phases covered by the duration of each session are calculated, and a histogram is plotted. I tested the assumption that a uniform phase distribution positively affects the localizability of the minimum. The deviation from the uniform distribution was evaluated through the Kolmogorov–Smirnov test. No correlation between the localizability of the minimum and the uniformity of phase distribution was found.

Several different approaches were applied to examine the limitations of the inverse method. The ecliptic poles were found with  $RMS_{min}$  in tenths of milimagnitude. To better determine the limiting dependency between the spatial distribution and the localizability of the ecliptic pole, more computation for more objects is necessary. As for the time distribution examination, a different approach would be needed to find a correlation. As for the number of points in time, rather than the observation of dense light curves, measurements that are sparse in time are more time efficient and fully sufficient, as was shown by Kaasalainen (2004). The future work would be to create a larger data set and to go into a lot more detail. An ambitious objective could be to derive a function combining all the studied parameters and returning a particular verdict on whether such a data set is sufficient enough.

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# Appendix

## **Spatial distribution**



Figure 6.33: The spatial distribution for 28 input session of (29) Amphitrite and the heatmap obtained through the light curve inversion method using these input sessions. The figure in the middle shows the complete heatmap. The *x* axis shows  $\lambda$  in degrees, the ecliptic pole coordinates changed with a step of 5°. The figure on the right shows only configuration of the ecliptic pole whose RMS differed from RMS<sub>min</sub> by 20 per cent of RMS<sub>max</sub> – RMS<sub>min</sub> at maximum. The rest of the values were left RMS = 20 mmag for visibility purposes. The minimal RMS<sub>min</sub> = 10.91 mmag is located in the box with coordinates  $\lambda = 325^{\circ}$ ,  $\beta = -30^{\circ}$ .



Figure 6.34: The spatial distribution for 28 input session of (29) Amphitrite and the heatmap obtained through the light curve inversion method using these input sessions. The minimal  $\text{RMS}_{\text{min}} = 9.79 \text{ mmag}$  is located in the box with coordinates  $\lambda = 330^{\circ}$ ,  $\beta = -30^{\circ}$ .



Figure 6.35: The spatial distribution and the heatmap for 28 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 13.72 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}, \beta = -20^{\circ}$ .



Figure 6.36: The spatial distribution and the heatmap for 20 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 13.70 \text{ mmag}$  is located in the box with coordinates  $\lambda = 120^{\circ}, \beta = -20^{\circ}$ .



Figure 6.37: The spatial distribution and the heatmap for 20 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.60 \text{ mmag}$  is located in the box with coordinates  $\lambda = 325^{\circ}, \beta = -30^{\circ}$ .



Figure 6.38: The spatial distribution and the heatmap for 20 input session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 12.39 \text{ mmag}$  is located in the box with coordinates  $\lambda = 25^{\circ}, \beta = -40^{\circ}$ .

# Spatial distribution for input sessions with more than 15 points



Figure 6.39: The spatial distribution for 10 input session of (29) Amphitrite and the heatmap obtained through the light curve inversion method using these input sessions. The input sessions have more than 15 points. The minimal RMS<sub>min</sub> = 9.41 mmag is located in the box with coordinates  $\lambda = 140^{\circ}$ ,  $\beta = -10^{\circ}$ .



Figure 6.40: The spatial distribution and the heatmap for 10 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 10.58 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -15^{\circ}$ .



Figure 6.41: The spatial distribution and the heatmap for 15 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 10.85 \text{ mmag}$  is located in the box with coordinates  $\lambda = 130^{\circ}$ ,  $\beta = -20^{\circ}$ .



Figure 6.42: The spatial distribution and the heatmap for 15 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 8.19 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -15^{\circ}$ .



Figure 6.43: The spatial distribution and the heatmap for 15 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 10.02 \text{ mmag}$  is located in the box with coordinates  $\lambda = 325^{\circ}$ ,  $\beta = -25^{\circ}$ .



Figure 6.44: The spatial distribution and the heatmap for 20 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.59 \text{ mmag}$  is located in the box with coordinates  $\lambda = 320^{\circ}$ ,  $\beta = -25^{\circ}$ .



Figure 6.45: The spatial distribution and the heatmap for 20 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.68 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -15^{\circ}$ .

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Figure 6.46: The spatial distribution and the heatmap for 20 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.12 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -20^{\circ}$ .



Figure 6.47: The spatial distribution and the heatmap for 25 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.50 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -15^{\circ}$ .



Figure 6.48: The spatial distribution and the heatmap for 25 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.26 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -20^{\circ}$ .

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Figure 6.49: The spatial distribution and the heatmap for 25 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 12.13 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -20^{\circ}$ .



Figure 6.50: The spatial distribution and the heatmap for 30 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 9.78 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -25^{\circ}$ .



Figure 6.51: The spatial distribution and the heatmap for 30 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 9.63 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -20^{\circ}$ .



Figure 6.52: The spatial distribution and the heatmap for 30 (no low-point) session of (29) Amphitrite. The minimal  $\text{RMS}_{\text{min}} = 11.09 \text{ mmag}$  is located in the box with coordinates  $\lambda = 135^{\circ}$ ,  $\beta = -25^{\circ}$ .



### **Time distribution**

Figure 6.53: The histograms showing the time distribution of (29) Amphitrite.



Figure 6.54: The histograms showing the time distribution of (29) Amphitrite.



Figure 6.55: The histograms showing the time distribution of (29) Amphitrite.



Figure 6.56: The histograms showing the time distribution of (29) Amphitrite.

#### A showcase of written Python routines

Listing 6.1: Python code to calculate and evaluate the time distribution of chosen sets. The outcomes are a histogram of the time distribution, the plot comparing the empiric distribution function of the dataset with the cumulative distribution function of uniform distribution and the statistic values of the Kolmogorov-Smirnov test.

```
import os
import glob
from scipy import stats
import numpy as np
import matplotlib.pyplot as plt
import math
# cumulative distribution function of uniform distribution
def phi(x):
    y = np.zeros(len(x))
    a = 0
    b = 1
    u = 1/(b-a)
    for k in range(len(x)):
        if (x[k] \le a):
             y[k] = 0
        elif (x[k] > a \text{ and } x[k] < b):
             y[k] = u * x[k]
        else:
             y[k] = 1
    return y
# per = sys.argv[1]
per_h = 5.138238 # rotational period in hours
per = per_h/24
bi = 20 # bins in the histogram
```

```
# searches the folder for all files to calculate for
path = os.path.abspath(\__file\__)
path = path.partition("time_guts.py")[0] + '*'
full_folder = glob.glob(path)
folder = []
for i in range(len(full_folder)):
    if '_lc_SPACE' in full_folder[i]:
        folder.append(full_folder[i])
# loads the folder with all sessions in database
seznam = []
with open('lc_tab.txt', 'r') as JD:
    for line in JD:
        seznam.append(line.split("\t"))
lc = int(seznam[0][0].strip("\n")) # number of sessions
# runs for each file
for j in range(len(folder)):
    name = folder[j]. partition ('Time_distribution \langle \rangle')[2]
    name = name.partition("_SPACE")[0]
    op = name+'_time_table.txt'
    print(name)
    # loads the file to know which sessions were used
    data = []
    i = 0
    with open(folder[j], 'r') as f:
        for line in f:
            data.append(line.split("t"))
    t = np.zeros([len(data),2])
    p = np.zeros(len(t))
    for i in range(len(data)):
        t[i][0] = data[i][0] # start time
        1 = 1
        # for all sessions in the file
        for k in range(lc):
            points = int(seznam[1][0])
            # was this session used
            if float(seznam[1+1][0]) = float(t[i][0]):
                 t[i][1] = float(seznam[1+points][0])
            1 = 1 + points + 1
```

```
T = t.min() \# find the starting time M_0
p = np.array(0.0) \# add the phase of M_0
b = np.abs(t[0][1] - T) / per
p = np.append(p, b - np.floor(b))
for i in range(1, len(t)): # for each session
    a = np.abs(t[i][0] - T) / per
    b = np.abs(t[i][1] - T) / per # the phase function
    aa = a - np.floor(a) # the phase
    bb = b - np.floor(b)
    h_a = int(np.floor(aa*bi)) # start bin
    h_b = int(np.floor(bb*bi)) # end bin
    # if it occupies only two boxes side by side
    if (h_a - h_b = 1.0 \text{ or } h_a - h_b + bi = 1.0)
        and (b-a < 1/bi):
        p = np.append(p,aa)
        p = np.append(p,bb)
    # edge crossing (e.g. 0.95 and 0.05)
    elif (aa>bb) and (b-a<1):
        p = np.append(p,aa)
        for k in range (h_a+1, bi):
            p = np.append(p,k/bi+0.5/bi)
        for k in range(h_b):
            p = np.append(p,k/bi+0.5/bi)
        p = np.append(p,bb)
    # if there is at least one full rotation
    elif (b-a \ge 1):
        # add +1 to bins from h_{-a}
        p = np.append(p,aa)
        for k in range (h_a+1, bi):
            p = np.append(p,k/bi+0.5/bi)
        # goes over the edge more than once
        if (b-a >=1+1/bi) and (aa>bb):
             dif = int(np.floor(b-a))
            for kk in range(dif): # add +1 to all bins
                 for k in range(bi):
                     p = np.append(p,k/bi+0.5/bi)
        # add +1 to bins up to h_b
        for k in range(h<sub>-</sub>b):
```

```
p = np.append(p,k/bi+0.5/bi)
        p = np.append(p,bb)
    # <1 rotation, no edge crossing
    else:
        p = np.append(p,aa)
        for k in range (h_a+1,h_b):
            p = np.append(p,k/bi+0.5/bi)
        p = np.append(p,bb)
# distribution functions
x = np.sort(p)
y = np.linspace(0, 1, len(x))
plt.figure()
plt.subplot(2,1,1)
plt.step(x,y,'b') # EDF of dataset
plt.plot(x, phi(x), 'r')
plt.subplot(2,1,2)
plt.plot(x,y-phi(x)) # CDF of uniform distribution
# Kolmogorov-Smirnov test
print('Limit_for_distance:',1.92/np.sqrt(len(x)))
stat , pvalue = stats . kstest(x, 'uniform')
stat, pvalue = stats.kstest(x, 'norm')
print('statistics:\t', stat)
print('p-value:\t', pvalue)
# plotting the histogram
plt.figure()
fig = plt.hist(p,bins=bi, range=(0.0, 1.0),
                density=False, rwidth = 0.9)
plt.title(name)
plt.ylabel('Number_of_occurencies')
plt.xlabel('Phase_in_rotational_period')
plt.xticks(np.arange(0.0,1.1,0.1))
```

Listing 6.2: and a ratio between the area covered by observation and the area of the ellipse.]Python code to calculate the basis of the orbit from data from Horizons System and the change-of-basis matrix. The change of basis transformation follows. The outcome is the 2D plot of the transformed points fitted with an ellipse [e7] and a ratio between the area covered by observation and the area of the ellipse.

```
import numpy as np
import math
import matplotlib.pyplot as plt
from scipy.spatial import ConvexHull
import sys
def fit_ellipse(x, y):
    function from [\citet{elipsa}]
    return np.concatenate((ak, T@ak)).ravel()
def cart_to_pol(coeffs):
    function from [\citet{elipsa}]
    return x0, y0, ap, bp, e, phi
def get_ellipse_pts (params, npts=100,
                     tmin=0, tmax=2*np.pi):
    function from [\citet{elipsa}]
    return x, y
file = sys.argv[1]
space = sys.argv[2]
# loads the output from Horizons System
data = []
i = 0
with open(file, 'r') as f:
    for line in f:
        data.append(line.split("_"))
        if line == '\ SOE \n': # start of the table
             start = i
        if line == '\ EOE \n': # end of the table
            end = i
        i = i + 1
N = int(end-start - 1)
# table with: time x y z
x = np.zeros(N)
y = np.zeros(N)
```

```
z = np.zeros(N)
t = np.zeros(N)
table = [[] for i in range(N)]
u = 0
1 = 1
for i in range(N):
    table[i] = [float(data[start+i+1][0].strip(',')),
                 float(data[start+i+1][4].strip(',')),
                 float (data [start+i+1][5]. strip ('.
                                                     , ')),
                 float (data [start+i+1][6]. strip (', n')]
    t[i] = table[i][0]
    x[i] = table[i][1]
    y[i] = table[i][2]
    z[i] = table[i][3]
# preparing for tranformation
# selects three points A B C to find the basis
sample = int(math.floor(N/3))
if N>4:
    A = np.array([table[sample][1], table[sample][2],
                    table [ sample ] [ 3 ] ])
    B = np. array ([table [2 * sample ][1], table [2 * sample ][2],
                    table [2* sample ] [3] )
    C = np. array ([table [3 * sample - 1][1]),
                    table[3*sample - 1][2],
                    table[3*sample - 1][3]])
else:
    A = np. array ([table [0][1], table [0][2], table [0][3]])
    B = np. array ([table [1][1], table [1][2], table [1][3]])
    C = np. array ([table [2][1], table [2][2], table [2][3]])
AB = np. array ([B[0]-A[0], B[1]-A[1], B[2]-A[2]])
AC = np. array ([C[0] - A[0], C[1] - A[1], C[2] - A[2]])
normal = np.cross(AB,AC)
\# x, y, and z basis vectors
o_x = AB/(np.sqrt(AB[0]**2+AB[1]**2+AB[2]**2))
o_z = normal/(np.sqrt(normal[0]**2+
                        normal[1]**2+normal[2]**2))
o_y = np.cross(o_x, o_z)
# change of basis
x_n = np.zeros(N) # new coordinates
y_n = np.zeros(N)
```

```
z_n = np.zeros(N)
matrix = np.array([o_x, o_y, o_z]) # change-of-basis matrix
for i in range(N):
    vec_old = np. array([x[i], y[i], z[i]])
    vec_new = matrix.dot(vec_old)
    x_n[i] = vec_new[0]
    y_n[i] = vec_new[1]
    z_n[i] = vec_new[2]
# saving of the 2D points
with open(space, 'w') as S:
    for i in range(N):
        text = str(t[i]) + ' \setminus t' + str(
             x_n[i] + ' t + str(y_n[i]) + ' n'
        S. write (text)
# fitting of an ellipse
coeffs = fit_ellipse(x_n, y_n)
print ('F(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0')
print ('a, _b, _c, _d, _e, _f =', coeffs)
x0, y0, ap, bp, e, phi = cart_to_pol(coeffs)
print ('x0, vy0, ap, bp, e, phi = ', x0, y0, ap, bp, e, phi)
el_x, el_y = get_ellipse_pts((x0, y0, ap, bp, e, phi))
name = file.partition('_OUT')[0]
# the area covered by the observations
pts1 = []
for i in range(N):
    pts1.append([x_n[i], y_n[i]])
pts = np.array(pts1)
hull = ConvexHull(pts)
plt.figure(5)
plt.fill(pts[hull.vertices,0], pts[hull.vertices,1],
          'red', alpha = 0.5)
plt.plot(el_x, el_y, 'gray')
plt.plot(x_n, y_n, 'o')
plt.title(name)
plt.show()
# the ratio between the areas
area = np.pi * ap* bp
print ("Area_between_points_=t", hull.volume)
print ("Area  of  ellipse \setminus t = t", area)
print ("Ratiot t t = t", hull.volume/area)
```

## Heatmaps of other objects

During the search for an object with only one minimal RMS area (without the effects of the ambiguity theorem described in Kaasalainen & Lamberg (2006), numerous other heatmaps were computed. The results of these computations are in this section.





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