

Lecture notes on stellar atmospheres

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Foreword

This document contains lecture notes on the course MNFFY X1/H98 “Stjerneatmosfærer” (Stellar atmospheres) read in the fall semester 1998 at the Physics Institute of NTNU, Trondheim. The course follows the classical text-book “Stellar atmospheres” by D. Mihalas, [3]. Only comments and additions to chapters (listed according to the 2nd edition) included into the program of the course will be explicitly written in the present document. Some more topics connected mainly with the observational practice (spectroscopy and photometry) and with the cases of extended atmospheres and atmospheres of binaries are added by the lecturer. The course could be continued on a summer practice at Ondřejov observatory close to Prague (at the 2-m spectroscopic telescope) and there are different possibilities for master or PhD. thesis in both theoretical as well as observational branch of the field.

The Chapter 5 has been added for the purpose of the course SIF4086 “Stjernefysikk” (Physics of stars) read in the fall semester 2000 following the text-book “The physics of stars” by A.C. Phillips, [5].

Chapter 1

Introduction

1.1 Goals of the stellar-atmosphere studies

Theory of stellar atmospheres is important for manifold reasons. First, by their definition, stellar atmospheres are the surface layers of stars where the observable spectrum is formed. The detailed understanding of the physical processes influencing the formation of outgoing light in this region is thus crucial for its quantitative analysis and gaining an information from the observations. This facilitates our knowledge of basic stellar parameters and the inner structure of stars. In this way, the theory of stellar atmospheres contributes to the progress of many other branches of astrophysics up to cosmology.

Another importance of the subject in question consists in its physical background. The basic feature of stellar atmospheres is their thermodynamic non-equilibrium which is dominated by the strong anisotropic radiation field. Stellar atmospheres can thus be used as laboratories of plasma physics, in which the atomic parameters (probabilities of quantum transitions) can be tested.

Also the mathematical methods developed in stellar atmospheres can be widely used in many other problems.

1.2 Basic concepts

Taking the atmosphere of the Sun as the best known example, we find its thickness of the order of several hundreds of km (cf. [3, chap. 7-7]), which is well negligible in comparison with the solar radius. This explains why we can see the sharp edge of the limb of solar disk and also why in the standard models we mostly approximate stellar atmospheres by plane-parallel slabs. From the H-R diagram we can estimate the range of effective temperatures ($\sim 3 \times 10^3 - 5 \times 10^4 \text{K}$) and gravity accelerations ($\sim 10^4 - 10^5 \text{g/cm}^2$ for main sequence stars and $\sim 1 - 3 \times 10^3 \text{g/cm}^2$ for supergiants), which are the main characteristic parameters of any atmosphere. It follows from here, that for hot supergiants the thickness of their atmospheres is not negligible and spherical models give better approximation. The solar observations do not in fact justify the second part of the standard assumption, i.e., the homogeneity in the plains of symmetry of stellar atmospheres. Spots with different effective temperatures (and sometimes also chemical composition and magnetic field) are observed in even larger extend in some other stars. Homogeneity of atmospheres is violated also by gravity darkening in rotating stars or binary components. These effects are usually neglected or, in the better case, modelled by a mosaic composed of locally homogeneous models with varying parameters. There are also other effects mostly ignored in the standard plane-parallel model, like

the anisotropy caused by either the magnetic field or by the motion of the plasma due to the stellar wind or some oscillations.

Taking into account the typical mass of stellar atmospheres, which is of the order of g/cm^2 and consequently it is even more negligible with respect to the rest of star than its volume, the problem of modelling the atmospheres may seem to be childish compared to the models of stellar interiors. This impression may be strengthened by the facts that the energy radiated by stars is released by thermonuclear reactions in stellar interiors, while in the atmospheres it is only reradiated, and that the interiors are modelled in sequences of the time evolution, while for the atmospheres is calculated their instantaneous picture only. However, the fact of thermodynamic non-equilibrium mentioned in the previous Section makes the physics and especially the numerical modelling of atmospheres more complicated. For modelling of the instantaneous radial structure of stellar interiors we solve a set of ordinary differential equations and the evolution in the time-coordinate is simply integrated as causal consequence of the last previous model. However, for the model atmosphere (even with the above mentioned simplifying assumptions) we need to solve a set (in frequencies) of two-dimensional integro-differential equations for the intensity of light as a function of depth and angle. These equations are mutually coupled with other equations of the structure of atmosphere.

The thermodynamic non-equilibrium of the radiation in stellar atmospheres can be in fact verified by the simple observation with naked eye – the fact that we can see stars shining on the background of the sky implies the existence of macroscopic flows, which should disappear in an exact equilibrium. While the same radiative energy flow in the interiors is only a small perturbation of the very dense radiation field, in the atmospheres we meet at one point outflowing radiation with distribution corresponding approximately to the effective temperature of the star and almost no radiation in the opposite direction. There is thus great discrepancy between the colour temperature of the radiation and the temperature which corresponds to the mean (in all directions) of the density of light. Different radiative processes in the plasma which tend to establish thermodynamic equilibrium between the inner degrees of freedom (kinetic, ionization or excitation energy levels) and the radiation thus pull the relative temperatures¹ of different couples of states to different values. This source of non-equilibrium is partly compensated by non-radiative (collisional) transitions which tend to establish the equilibrium of the inner states. The competition of these two processes leads to a statistical equilibrium, which is close to thermodynamic equilibrium between states for which the collisions prevail, while for states strongly interacting with the radiation there can be significant deviations from the equilibrium distribution (corresponding to the temperature of other states).

¹Let us remind that the temperature can be exactly defined only for states which are at thermodynamic equilibrium, i.e., which obey an equilibrium distribution law (e.g., Boltzmann distribution for classical particles or Bose-Einstein or Fermi-Dirac for quantum particles) with the corresponding temperature. For two energy levels we can find a temperature describing the ratio of their population. This temperature may be even negative, which facilitates a laser effect.

Chapter 2

Radiative transfer

This chapter corresponds to chapters 1, 2, 3 and 6 of [3].

2.1 Equation of radiative transfer

In classical approximation (i.e., not quantum-field view), the radiation field can be described either phenomenologically using the specific intensity, or in the kinetic approximation as a gas of photons, or as a superposition of electromagnetic waves.¹ Unlike the Mihalas' Eq. (1-2), it is more advantageous in the kinetic description to use the distribution function f normalized per unit quantum state² $\frac{d^3x d^3p}{h^3}$, where the momentum of photon

$$\vec{p} = \frac{h\nu}{c} \vec{n}, \quad (2.1)$$

and hence

$$d^3p = \frac{h^3}{c^3} \nu^2 d\nu d^2\omega, \quad d^3x = c dt d^2S. \quad (2.2)$$

The specific intensity I is thus related to f as

$$I = \frac{2h\nu^3}{c^2} f, \quad (2.3)$$

where the factor 2 corresponds to possible polarization states of photons. Polarization is usually neglected in models of stellar atmospheres, despite it is known that it can cause an error of several percents of the intensity even in plane-parallel atmosphere dominated by Thomson scattering. A

¹Cf. [3, pp. 7-9] – The choice of units (SI, CGSE or CGSM) influences the coefficients in equations of electromagnetic field and must be carefully watched if the effects of electromagnetic field on the radiating plasma are investigated (e.g., the magnetohydrodynamics of the atmosphere or the Zeeman effect in line profiles). However, the present aim is simply to demonstrate the consistency of the description of radiative field and its stress-energy tensor in both the particle- and field- approach.

²The main advantage of the normalization per unit element of phase-space volume consists in the Lorentz-invariance of the phase-space element and thus also of f . It can be proved in relativistic kinetic theory, however, intuitively it is obvious from the Lorentz contraction of the space-volume, which is inversely proportional to relativistic transformation of momentum. Unlike f , the specific intensity I , which includes the energy of photons and which is normalized per element of spherical coordinates $d^2\omega d\nu$, is thus influenced by Doppler-shift and aberration in treatment of radiative transfer in differentially moving media. The choice of quantum state as the unit of phase-space makes simpler the expressions for probabilities of quantum transitions and also the quantum statistics; the Planck's black-body radiation is described by the simple Bose-Einstein distribution $f = (\exp(\frac{h\nu}{kT}) - 1)^{-1}$.

more sophisticated formulation of radiative transfer can be done in terms of Stoke's parameters describing the partial linear as well as circular polarization of the light.

If the distribution function f of photons or the specific intensity I are known, the basic properties of the radiation field can be calculated. The most important are the first three moments of the distribution, which are the components of the stress-energy tensor, i.e., the density of the energy of radiation

$$\varepsilon_r = 2 \int h\nu f \frac{d^3p}{h^3} = \frac{1}{c} \int I_\nu d\nu d^2\Omega, \quad (2.4)$$

the density of momentum π_r^i or the density of the flow of radiative energy

$$F_r^i = c^2 \pi_r^i = 2 \int h\nu n^i c f \frac{d^3p}{h^3} = \int n^i I_\nu d\nu d^2\Omega, \quad (2.5)$$

and the stress tensor (pressure) of the radiation

$$T_r^{ij} = 2 \int h\nu n^i n^j f \frac{d^3p}{h^3} = \frac{1}{c} \int n^i n^j I_\nu d\nu d^2\Omega. \quad (2.6)$$

In the kinetic view of the radiation field, the equation of radiative transfer is the Boltzmann equation for the gas of photons

$$\frac{d}{dt} f(t, x, p) \equiv \frac{\partial}{\partial t} f + \frac{dx^i}{dt} \frac{\partial}{\partial x^i} f + \frac{dp^i}{dt} \frac{\partial}{\partial p^i} f = \left(\frac{\delta f}{\delta t} \right)_c, \quad (2.7)$$

which is, generally, a consequence of the Liouville theorem, i.e., the conservation of phase-space volume. This is valid for any Hamiltonian motion. The phase-space density f of particles is then constant for collision-free particles, or its change is proportional to the collisional term $(\frac{\delta f}{\delta t})_c$ on the right-hand side of the Boltzmann equation (2.7). This term gives the number of particles which will appear in the unit phase-space volume per unit time due to the collisions (minus the number of particles which will disappear). Changing the parametrization to the length $s = ct$ measured along the ray and substituting for f from Eq. (2.3) in terms of the specific intensity I , the equation of radiative transfer reads³

$$\frac{d}{ds} I(t, x, n, \nu) = \frac{\partial}{c\partial t} I + \frac{dx^i}{ds} \frac{\partial}{\partial x^i} I + \frac{dn^i}{ds} \frac{\partial}{\partial n^i} I + \frac{d\nu}{ds} \nu^3 \frac{\partial}{\partial \nu} \nu^{-3} I = \left(\frac{\delta I}{\delta s} \right)_c. \quad (2.8)$$

In the case of photons in Cartesian coordinates in a non-refractive medium, the momentum (i.e., both the direction \vec{n} and the frequency ν) is constant and $\frac{dx^i}{ds} = n^i$. Usually we deal with a time-independent problem only, so that the term with the time-derivative also vanishes and the equation of radiative transfer thus reduces to its standard form

$$\frac{d}{ds} I(t, x, n, \nu) = n^i \frac{\partial}{\partial x^i} I = \left(\frac{\delta I}{\delta s} \right)_c. \quad (2.9)$$

However, in curvilinear coordinates (e.g., spherical) or in the case of refraction or gravitational bending of light, the change of n^i must also be included. In the case of the change of ν , e.g., due to the Doppler shift in a differentially moving medium or due to the gravitational redshift, the term with $\frac{\partial}{\partial \nu}$ must also be included (but it can not be commuted with the multiplier ν^3 from Eq. (2.3)).

³Note, that only two components of the unit vector \vec{n} are algebraically independent. This is why the angles of spherical coordinates in the p -space are often used instead of the components.

The collisional term on the right-hand side of Eq. (2.8) gives now the amount of energy radiated into the beam of light per its unit length (into its unit cross-section in unit solid angle and unit frequency interval). We can suppose that it is a sum of two terms

$$\left(\frac{\delta I}{\delta s}\right)_c = \epsilon - \alpha I, \quad (2.10)$$

the emission (ϵ) and the absorption⁴ ($-\alpha I$). The later is negative and proportional to the intensity of light. The absorption coefficient (or opacity) α is thus the probability that a photon from the beam will be absorbed in a unit length and it is proportional to the volume density of the absorbers multiplied by their effective cross-section. We usually replace the emission coefficient (emisivity) ϵ by the source function S defined by the relation

$$\epsilon = \alpha S. \quad (2.11)$$

The equation of radiative transfer thus reads

$$\frac{d}{ds}I(t, x, n, \nu) = \alpha(S - I). \quad (2.12)$$

There exists also the stimulated (induced) emission, which is positive and proportional to the intensity. For the purpose of solution of the equation of radiative transfer (2.12), we use to include it into the linear absorption term as a negative absorption. As it will be explained in Chapter 3, the stimulated emission is a consequence of the boson nature of photons, and it thus takes place in all radiative processes including both the true absorption/emission as well as all kinds of scattering.⁵

2.2 Formal solution of radiative transfer

The equation of radiative transfer along the ray (in terms of the optical depth τ , where $d\tau = \alpha ds$),

$$\frac{d}{d\tau}I = S - I, \quad (2.13)$$

has the formal solution

$$I(\tau) = \int_{-\infty}^{\tau} S(t)e^{-(\tau-t)} dt, \quad (2.14)$$

which can be interpreted as the superposition of exponentially damped contributions S of light radiated at depths $t < \tau$ of the ray (τ is increasing in direction of the ray in this case).

For the radiation field in a plane-parallel geometry the individual rays are parametrized by direction cosine μ and the formal solution of the equation of radiative transfer (with τ measured in the direction $\mu = -1$)

$$\mu \frac{\partial}{\partial \tau} I = I - S \quad (2.15)$$

thus reads

$$I(\tau, \mu) = \int_{\tau}^{\tau_b(\mu)} S(t, \mu) e^{-\frac{t-\tau}{\mu}} d\frac{t}{\mu}, \quad (2.16)$$

⁴Note that [3] uses the letters η instead of ϵ and χ instead of α .

⁵The notice in [3, p. 24] about its existence for a definit upper state only is thus wrong. In the opposite case it would lead to violation of the 3rd law of thermodynamics.

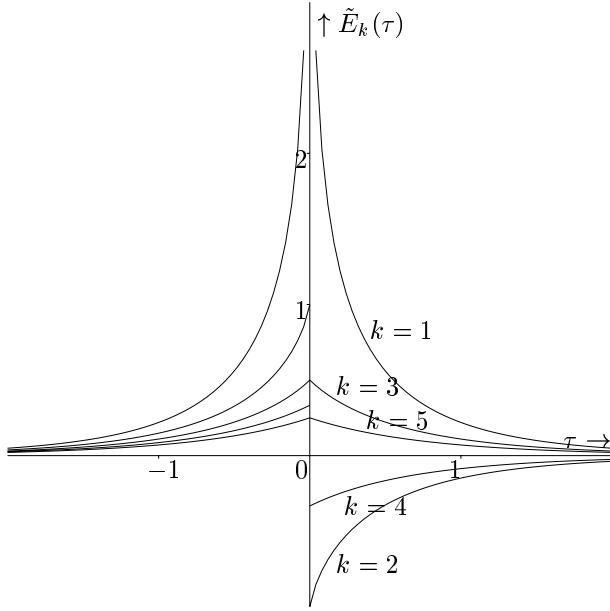


Figure 2.1: Modified exponential integral.

where the boundary optical depth $\tau_b(\mu) = \infty$ for outgoing rays, $\mu > 0$, and $\tau_b(\mu) = 0$ for in-going rays, $\mu < 0$ (cf. [3, Eqs. (2-50,51)]). The source function S is usually supposed to be isotropic ($\frac{\partial S}{\partial \mu} = 0$). However, it need not be the case, e.g., if there is an anisotropic scattering. Let us thus suppose that it can be expressed in the form of power-law series, $S = \sum_{l=0}^{\infty} S_l \mu^l$. The formal solutions (cf. [3, Eqs. (2-57,59,60)]) for moments $J \equiv M_0$, $H \equiv M_1$, $K \equiv M_2$ (i.e., up to the coefficients 4π or $4\pi c$ the components of the stress-energy tensor – cf. Eqs. (2.4), (2.5) and (2.6)), and generally any moment can thus be generalized and simplified as

$$\begin{aligned}
M_k(\tau) &\equiv \int_{-1}^1 I(\tau, \mu) \mu^k \frac{d\mu}{2} = & (2.17) \\
&= \sum_l \int_{-1}^1 \int_{\tau}^{\tau_b(\mu)} S_l(t) \mu^{k+l} e^{-\frac{t-\tau}{\mu}} d\frac{t}{\mu} \frac{d\mu}{2} = \\
&= \sum_l \int_{\infty}^0 S_l(t) \int_0^1 \mu^{k+l-1} \text{sign}^{k+l-1}(t-\tau) e^{-\frac{|t-\tau|}{\mu}} \frac{d\mu}{2} dt = \\
&= \sum_l \frac{1}{2} (S_l * \tilde{E}_{k+l+1}), & (2.18)
\end{aligned}$$

if we define $S_l(\tau) = 0$ for $\tau \leq 0$ to hold the convolution $*$ in the variable τ defined in the standard range of integration $(-\infty, \infty)$. Here we have introduced a modified exponential integral

$$\tilde{E}_k(\tau) = \text{sign}^{k-1}(-\tau) E_k(|\tau|), \quad (2.19)$$

which is defined in the whole range of real numbers and which differs from the standard exponential

integral

$$E_k(x) = x^{k-1} \Gamma(1-k, x) \quad (2.20)$$

$$= \int_0^1 \mu^{k-2} \exp\left(-\frac{x}{\mu}\right) d\mu \quad (2.21)$$

$$= \int_1^\infty y^{-k} \exp(-xy) dy \quad (2.22)$$

(defined for positive numbers) by the sign-convention only. It is obvious that the convolution with functions \tilde{E}_1 , \tilde{E}_2 and \tilde{E}_3 gives the kernels of integral operators Λ , Φ and X , resp. (cf. [3, Eqs. (2-58,61,62)]). The discontinuity of \tilde{E}_{2k} at zero results in δ -function in its derivative,

$$\frac{d}{d\tau} \tilde{E}_k(\tau) = \tilde{E}_{k-1}(\tau) - \frac{1 + (-1)^k}{k-1} \delta(\tau). \quad (2.23)$$

Applying this relation between operators to the source function one arrives at an infinite set of differential equations for the moments

$$\frac{d}{d\tau} M_k(\tau) = M_{k-1}(\tau) - \sum_l \frac{1 - (-1)^{k+l}}{2(k+l)} S_l, \quad (2.24)$$

which can also be obtained as moments of the transfer equation (2.15) (i.e., multiplying it by μ^{k-1} and integrating by $d\mu$), and which in the special case of isotropic S reduces Eqs. (2-71) and (2-79) of [3]

$$\frac{d}{d\tau} H = J - S, \quad (2.25)$$

$$\frac{d}{d\tau} K = H. \quad (2.26)$$

These equations can also be obtained by the integration of Eq. (2.15) with $d\mu$ or $\mu d\mu$, resp. They describe the conservation of radiative energy and momentum, which is important for the vertical structure of the atmosphere. For the total balance of energy and momentum in all frequencies these equations must be expressed in a frequency independent radial coordinate, e.g., z , and integrated in frequencies

$$\frac{d}{dz} \int H d\nu = \int \alpha(S - J) d\nu, \quad (2.27)$$

$$\frac{d}{dz} \int K d\nu = - \int \alpha H d\nu. \quad (2.28)$$

The first one states that the divergence of the radiative flux is the difference between the density of energy emitted and absorbed. In radiative equilibrium we suppose this difference to be zero. The second one gives the gradient of the pressure of radiation as the density of momentum transferred from the radiative flux to the matter by the absorption.

The formula (2.23) rewritten in the form

$$\tilde{E}_k(\tau) = \frac{d}{d\tau} \tilde{E}_{k+1}(\tau) + \frac{1 - (-1)^k}{k} \delta(\tau), \quad (2.29)$$

can be useful for practical calculations. For instance, in the case of a polynomial source function $S(\tau, \mu) = h(\tau)S_{lm}\mu^l\tau^m$ (h being the Heaviside function), we find⁶

$$\begin{aligned}
M_k &= \frac{1}{2}S_{lm}(h\tau^m) * \tilde{E}_{k+l+1} = \frac{1+(-1)^{k+l}}{2(k+l+1)}S_{lm}h\tau^m + \frac{1}{2}S_{lm}(h\tau^m)' * \tilde{E}_{k+l+2}(\tau) = \\
&= \frac{1+(-1)^{k+l}}{2(k+l+1)}S_{lm}h\tau^m + \frac{1}{2}S_{lm}\delta_0^m\tilde{E}_{k+l+2}(\tau) + \frac{1}{2}S_{lm}(mh\tau^{m-1}) * \tilde{E}_{k+l+2}(\tau) = \\
&= \frac{1}{2}S_{lm}h \left(\frac{1+(-1)^{k+l}}{k+l+1}\tau^m + \frac{1-(-1)^{k+l}}{k+l+2}m\tau^{m-1} + \dots \right) \\
&\quad + \frac{1}{2}S_{lm} \left(\delta_0^m\tilde{E}_{k+l+2}(\tau) + m\delta_0^{m-1}\tilde{E}_{k+l+3}(\tau) + \dots \right) . \tag{2.30}
\end{aligned}$$

In the special case of isotropic S linear in τ we thus arrive at Eqs. (2-63,64,65) of [3]. Recurrently, one can obtain from Eq. (2.29) also the expression for the modified exponential integrals as series of derivatives of δ -functions⁷

$$\tilde{E}_k(\tau) = \sum_{n=0}^{\infty} \frac{1-(-1)^{k+n}}{k+n} \frac{d^n}{d\tau^n} \delta(\tau) . \tag{2.31}$$

Applying this relation to the source function, we can express all moments as combinations of its derivatives

$$M_k(\tau) = \sum_{nl} \frac{1+(-1)^{k+l+n}}{2(k+l+n+1)} \frac{d^n}{d\tau^n} S_l , \tag{2.32}$$

i.e., in the special case of isotropic S we obtain Eqs. (2-89a,b,c) of [3]. and for the linear S also Eqs. (2-63,64,65) of [3].

Exercise 1 (a) Find the behavior of $\tilde{E}_k(x)$ at $x \rightarrow 0$ and $x \rightarrow \pm\infty$, (b) prove Eqs. (2.23) and (2.31), (c) prove the algebraic recurrent formula for $\tilde{E}_k(x)$,

$$\tilde{E}_{k+1}(x) = \frac{1}{k} \left(\text{sign}^k(-x)e^{-|x|} + x\tilde{E}_k(x) \right) .$$

Exercise 2 Derive the equation of radiative transfer and the corresponding moment equations for the case of spherical symmetry.

2.3 Problems of scattering and grey atmosphere

In the formal solutions (2.16) or (2.18) of the transfer equation (2.15) we supposed the source function $S(\tau, \mu)$ to be given (for each frequency) independently on the solution of radiation field. However, the emissivity of a coherent scattering is proportional to the specific intensity of the light,

⁶Here we use the property (A.4) of the convolution and the relation $(h\tau^m)' = \delta_0^m\delta(\tau) + mh\tau^{m-1}$.

⁷The convergent expansion of E_k reads

$$E_k(x) = \frac{(-x)^{k-1}}{(k-1)!} \left(-\ln x - \gamma + \sum_{m=1}^{k-1} \frac{1}{m} \right) - \sum_{m=0, m \neq k-1}^{\infty} \frac{(-x)^m}{(m-k+1)m!} ,$$

where the Euler's constant $\gamma \simeq 0.5772156649$.

e.g., for the isotropic scattering $\epsilon_s = \sigma J$. If the emissivity of standard absorption and emission $\epsilon_a = \kappa B$ (cf. [3, Sec. 2-1]), the total source function

$$S = \frac{\kappa B + \sigma J}{\kappa + \sigma} = (1 - \varsigma)B + \varsigma J, \quad (2.33)$$

where $\varsigma = \frac{\sigma}{\kappa + \sigma}$ is the so-called single scattering albedo. Eq. (2.15) of radiative transfer is thus differential in τ and integral in μ

$$\mu \frac{\partial}{\partial \tau} I = I - \varsigma \int_{-1}^1 I \frac{d\mu}{2} - (1 - \varsigma)B. \quad (2.34)$$

The formal solution (2.18) for the moments of radiation field, which in the present simplified case of isotropic source function reads for the mean intensity $J \equiv M_0$

$$J = \frac{1}{2}(S * \tilde{E}_1) \equiv \Lambda(S) = \Lambda[(1 - \varsigma)B] + \Lambda[\varsigma J], \quad (2.35)$$

is thus an operator equation⁸

$$J - \Lambda[\varsigma J] = \Lambda[(1 - \varsigma)B], \quad (2.36)$$

for J . This equation can be formally solved as

$$J = [1 - \Lambda\varsigma]^{-1} \Lambda[(1 - \varsigma)B], \quad (2.37)$$

however, to find the inverse operator $[1 - \Lambda\varsigma]^{-1}$ is not, generally, a simple task. In the case of small relative values ς of the scattering, the emissivity of scattering can be treated as a small perturbation, for which the correction can be found iteratively by the so called Λ -iteration

$$J_{(0)} = \Lambda[(1 - \varsigma)B], \quad J_{(n+1)} = J_{(0)} + \Lambda\varsigma J_{(n)}, \quad (2.38)$$

which is equivalent to the expression

$$[1 - \Lambda\varsigma]^{-1} = \sum_{n=0}^{\infty} [\Lambda\varsigma]^n, \quad (2.39)$$

for the function of the operator $\Lambda\varsigma$ in terms of the sum of geometric series. Unfortunately, in practice, the value of ς is close to 1, for which values the Λ -iteration converges very slowly.

Approximate Λ -iterations. The problem of convergence of Λ -iterations can be overcome (or at least decreased) in numerical calculations by the following method. If we denote the right-hand side of Eq. (2.36) as R and we split the Λ -operator into two parts

$$\Lambda\varsigma = A + (\Lambda\varsigma - A), \quad (2.40)$$

Eq. (2.36) reads

$$(1 - A)J - (\Lambda\varsigma - A)J = R, \quad (2.41)$$

⁸This equation is the Fredholm equation of the second kind. It can be alternatively written for unknown functions S or $J - B$,

$$S - \varsigma\Lambda(S) = (1 - \varsigma)B, \quad [1 - \Lambda\varsigma](J - B) = \Lambda B - B.$$

The operator A is chosen to approximate $\Lambda\zeta$, however, at the same time to be sufficiently simple so that $1 - A$ could be easily inverted. The equation for J can then be rewritten into the form

$$J - (1 - A)^{-1}(\Lambda\zeta - A)J = (1 - A)^{-1}R, \quad (2.42)$$

which again is formally similar to Eq. (2.36), but the operator $(1 - A)^{-1}(\Lambda\zeta - A)$ is now smaller and the iteration process of the type (2.39) converges much faster and in a wider region of ζ . This is why the abbreviation ‘ALI’ of the method is also interpreted as ‘accelerated Lambda-iteration’. The simplest choice of A is $A = \zeta \times 1$, i.e., the first term in the expression (2.29) for \tilde{E}_1 , however, the method is much more efficient for a three-diagonal matrix A in the representation in discrete values of optical depth.

Grey atmosphere. In the respect of the above discussed problems with convergence in a scattering atmosphere, the extreme case is that of a grey atmosphere in radiative equilibrium. The assumption of greyness consists in frequency independence of the total (absorption plus scattering) opacity α . The transfer equation (2.15) can thus be integrated in frequency and it has the same form for the bolometric intensity and also its moments (2.27), (2.28) etc. are identical with (2.25), (2.26) resp. Due to the additional assumption of radiative equilibrium we thus have

$$\frac{d}{d\tau}H = J - S = 0 \quad (2.43)$$

for the bolometric quantities, which thus behave as a pure scattering $\zeta = 1$ (despite the emission need not balance the absorption in individual frequencies). The equation of radiative transfer thus is the linear homogeneous equation

$$\mu \frac{\partial}{\partial \tau} I = I - \int_{-1}^1 I \frac{d\mu}{2}, \quad (2.44)$$

and its (nontrivial) solutions must be parameterized by the required value of the radiative flux (which is constant due to the radiative equilibrium). An approximate or exact solution of this equation can be found by different methods (cf. [3, Sec. 3-3, 3-4]). A simple but useful approximation is the Milne - Eddington approximation, which can be found by the following procedure. According to Eq. (2.26), the radiation pressure is exactly

$$K = H\tau + c \quad (2.45)$$

in the radiative equilibrium. We can define the Eddington factor

$$f \equiv \frac{K}{J} = \frac{\int \mu^2 I d\mu}{\int I d\mu}. \quad (2.46)$$

Due to its definition, $f \in (0, 1)$; f can be close to 0 if there prevails the radiation in the direction of the plain of symmetry. This could happen in a finite slab (which can model, e.g., an accretion disk). However, in a stellar atmosphere we expect almost isotropic radiation in deep layers, in which case $f = \frac{1}{3}$. Close to the surface the radiation should be more concentrated in directions with higher μ , because the corresponding beams are coming from deeper and hotter regions. f is thus higher than $\frac{1}{3}$ here. It could approach the value of 1 for the outward peaking of radiation $I(\mu) \sim \delta(\mu - 1)$ at large distances of spherical stars, however, for a planeparallel atmosphere only a moderate concentration to $\mu \sim 1$ can be expected. Note, that for a known dependence $f(\tau)$ (and $S(\tau)$), Eqs. (2.25) and (2.26) form a close set of equations for J and H , which is used in

some numerical methods, because f is of the order of 1 and it can thus be estimated without a large error. To fit properly the asymptotic behaviour at $\tau \rightarrow \infty$, we choose $f = \frac{1}{3}$ in the Milne – Eddington approximation, and we thus have

$$S = J = 3K = 3H(\tau + c) , \quad (2.47)$$

where c is any real integration constant. Following the formal solution (2.16),

$$I(\tau, \mu) = 3H(\tau + \mu + c) \quad (2.48)$$

for $\mu > 0$, and for $\mu < 0$ the intensity approaches this behaviour at large τ , where the missing radiation from above the surface is not perceivable. In fact, for any value of c , this is an exact solution of Eq. (2.44), however, it does not satisfy the required boundary conditions at the surface (which corresponds to the wrong values of f close to the surface). To minimize this disadvantage of the solution, we can fix c to the value, for which the source function (2.47) will give the correct value $H(\tau = 0)$ integrated with the proper boundary conditions, i.e., according to (2.18). Owing to the linear dependence of S on τ , we can use Eq. (2.30) and we find $c = \frac{2}{3}$. The exact solution of the grey atmosphere reads

$$S = 3H(\tau + q(\tau)) , \quad (2.49)$$

where the Hopf function q increases monotonically from $q(0) \simeq 0.577$ to $q(\infty) \simeq 0.710$, which agrees with the Milne – Eddington approximation quite well, at least in limits of applicability of the grey model to real atmospheres.

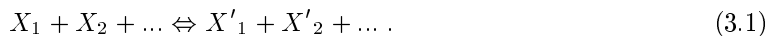
Chapter 3

Formation of the spectrum

This chapter corresponds to chapters 4 and 5 of [3].

3.1 Einstein coefficients

For practical calculations of radiative transfer one needs to know absorption, emission and scattering coefficients, i.e., the probabilities of radiative processes and the abundances of particles in states required for these processes. These abundances are influenced not only by the investigated radiative processes, but, generally, by all possible processes between the particles. These processes can be treated as some chemical reactions



From the phenomenological point of view, the rates of reactions per unit time and unit volume of space in each direction of the reaction are proportional (with constants characterizing the probability of the reaction) to the product of densities $n_1 n_2 \dots$ and $n'_1 n'_2 \dots$. However, the balance of the reaction can be described also on a more detailed level of transitions between particular elementary quantum states of both ingoing and outgoing particles (including, e.g. their kinetic degrees of freedom). In this case the product of densities should be replaced by the product of the distribution functions normalized per quantum state (i.e. the occupation numbers). If we assume the distributions into the states of the participating particles to be known, the macroscopic rates can be then obtained by the averaging of the elementary probabilities over the distribution of initial states and by summation over the final states of particles participating in the reaction. Hereafter, we shall thus investigate the reaction (3.1) as a transition between particular quantum states.

In the detailed equilibrium, the numbers of reactions in both directions are balanced,

$$f_1 f_2 \dots P_{(12\dots \rightarrow 1'2'\dots)} = f'_1 f'_2 \dots P_{(1'2'\dots \rightarrow 12\dots)} \quad (3.2)$$

For example, in the case of decay of a particle in an upper excited state X_u into a lower state X_l with simultaneous emission of another particle p ,

$$X_u \Leftrightarrow X_l + p, \quad (3.3)$$

the detailed balance

$$f_u P_{ul} = f_l f_p P_{lu} \quad (3.4)$$

(where P_{ul}, P_{lu} stand for the probabilities normalized per single states of u, l and p of the reaction in its corresponding directions), will occur, if the distribution of the particle p will be

$$f_p = \frac{f_u P_{ul}}{f_l P_{lu}}. \quad (3.5)$$

If the populations of both states $X_{u,l}$ are in thermodynamic equilibrium, i.e., if they satisfy the Boltzmann distribution $f_{u,l} \sim \exp(-E_{u,l}/kT)$, we thus find that the particle p carrying the difference between energies of both states, $E_p = E_u - E_l$, must obey the Boltzmann distribution corresponding to the same temperature,

$$f_p = \frac{P_{ul} \exp(-\frac{E_u}{kT})}{P_{lu} \exp(-\frac{E_l}{kT})} = \frac{P_{ul}}{P_{lu}} \exp(-\frac{E_p}{kT}). \quad (3.6)$$

It is thus obvious that this description is inappropriate just in the desired case of radiative processes, because the equilibrium distribution of photons is given by the Planck distribution instead of the Boltzmann one.

The difference between the behaviour of photons and the above described behaviour of the non-quantum particle p consists in their bosonic nature, which makes the rates of their reactions dependent not only on the densities of particles incoming to the reaction, but also on the densities (occupation numbers of final states) of the outgoing bosons (and fermions, if they also participate in the reaction). Consequently, if the particle p in the process (3.3) should be a photon, the above given classical description (3.4) of the statistical equilibrium must be modified by enhancing the left hand side for the contribution of the so called ‘induced emission’ (to distinguish it from the simple ‘spontaneous emission’), i.e., a process in which another photon (in the same state as the emitted one) participates. Let us denote the probability of spontaneous emission by one atom in the state u per unit time¹ into the chosen states l of the atom and p of the photon

$$P_{ul}^{\text{sp.}} = \frac{c}{h\nu} A_{ul}. \quad (3.7)$$

In addition to this spontaneous emission, the upper state can emit the photon by spontaneous emission with probability proportional to the occupation number of the emitted photon (due to its interaction with the already existing photon in this state),

$$f_p P_{ul}^{\text{ind.}} = \frac{c}{h\nu} I B_{ul}. \quad (3.8)$$

The process in opposite direction can occur only if the absorbed atom yields the energy needed for the excitation of the atom. Consequently, the probability of absorption (during a unit of time per atom in the l -state in a given radiation field) is proportional to the occupation number of photons,

$$f_p P_{lu} = \frac{c}{h\nu} I B_{lu}. \quad (3.9)$$

The equation of statistical balance of radiative processes, which should replace the non-quantum equation (3.4), thus reads

$$f_u A_{ul} + f_u I B_{ul} = f_l I B_{lu}. \quad (3.10)$$

¹Note that according to Eqs. (4-2) – (4-4), or their use in (4-10), in [3, p. 78] the Einstein coefficients A_{ul} , B_{ul} and B_{lu} are introduced as probabilities per light-travel time of unit length, not per the unit time, and they give the energy, not the number of emitted photons. The corresponding multiplier given by the speed and energy of photons cancels from the rate equations, however it must be included or skipped self-consistently in the treatment of radiative transfer as well as of the statistical equilibrium.

The detailed equilibrium of this reaction thus takes place if the distribution of photons given in terms of the specific intensity satisfies

$$I = \frac{f_u A_{ul}}{f_l B_{lu} - f_u B_{ul}} = \frac{A_{ul}}{B_{ul}} \frac{1}{\frac{f_l B_{lu}}{f_u B_{ul}} - 1} . \quad (3.11)$$

In thermodynamic equilibrium the states of the atom should be populated according to the Boltzmann distribution

$$\frac{f_l}{f_u} = \exp\left(-\frac{E_l - E_u}{kT}\right) = \exp\left(\frac{h\nu}{kT}\right) , \quad (3.12)$$

where we have used the relation $E_u - E_l = h\nu$ of conservation of the energy, and the intensity

$$I = \frac{A_{ul}}{B_{ul}} \frac{1}{\frac{B_{lu}}{B_{ul}} \exp\left(\frac{h\nu}{kT}\right) - 1} \quad (3.13)$$

should be equal to the Planck's black-body radiation

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \quad (3.14)$$

with the same temperature T . It is possible only if the Einstein coefficients (up to now treated as characteristics of three independent quantum processes) will satisfy the Milne – Einstein relations

$$A_{ul} = \frac{2h\nu^3}{c^2} B_{ul} , \quad (3.15)$$

$$B_{ul} = B_{lu} . \quad (3.16)$$

In practice, we often denote as one atomic level not a single quantum state (e.g. l, u) but the so-called ‘gross-state’, i.e. a set of sub-states with (approximately) equal energies (e.g. L, U). The probability of transitions between these gross-states depends generally on the inner distribution of population of the initial states, e.g.

$$P_{UL} = \frac{\sum_{ul} f_u P_{ul}}{\sum_u f_u} . \quad (3.17)$$

However, in thermodynamic equilibrium, the populations f_u are equal in the whole set U of states u , i.e. the population of the gross-state is

$$f_U \equiv \sum_u f_u = g_U f_u , \quad (3.18)$$

where g_U is the statistical weight of the level, and hence

$$P_{UL} = \frac{1}{g_U} \sum_{ul} P_{ul} . \quad (3.19)$$

To calculate the contribution of the process (3.3) to the collisional term (2.10) on the right-hand side of equation of radiative transfer, we must re-normalize the number of individual processes from the single atom to the unit of volume

$$\left(\frac{\delta I}{\delta s}\right)_c \equiv \epsilon - \alpha I = n_u A_{ul} + n_u I B_{ul} - n_l I B_{lu} , \quad (3.20)$$

where $n_{u,l}$ are the volume densities of the atom in the corresponding state. The terms here correspond to the spontaneous emission, induced emission and the true absorption. Comparing the absolute term and the term linear in I , we find (taking into account also the Milne – Einstein relations) the emissivity

$$\epsilon = n_u A_{ul} = \frac{2h\nu^3}{c^2} n_u B_{ul} , \quad (3.21)$$

and the opacity

$$\alpha = n_l B_{lu} - n_u B_{ul} = n_l B_{ul} \left(1 - \frac{n_u}{n_l}\right) , \quad (3.22)$$

in which the true absorption is reduced for the negative absorption, i.e. the induced emission. The term in brackets is equal to $(1 - \exp(-\frac{h\nu}{kT}))$ in thermodynamic equation. However, if the thermodynamic equilibrium between the population of the upper and the lower level is violated, the populations must be substituted explicitly. In the case of ‘negative temperature’, i.e. the inversion $n_u > n_l$ the absorption coefficient is negative and the radiative transfer leads to the laser-effect, i.e. an exponential grow of the intensity.

Exercise 3 *Derive the relations between the overall Einstein coefficients for gross-states of degenerated atomic levels and those corresponding to individual sub-levels. Derive the Milne – Einstein relations (3.15), (3.16) for them. Hint: Sum the rate equation (3.10) for all sublevels and require the same form of the overall equation.*

3.2 Interaction of quantum systems with radiation field

Let us investigate a particle (atom, ion or molecule) described by its Hamiltonian H_P with quantum eigenstates $|a\rangle$,

$$H_P|a\rangle = E_{P,a}|a\rangle , \quad (3.23)$$

cf. e.g., [1, Chap. X]. The eigenvalues $E_{P,a}$ may be either in a discrete or in a continuous spectrum of the energy (the later case corresponds in fact to a system of particles, e.g., ion + electron).

The state of one photon is described by the momentum \vec{p} and polarization $\vec{\epsilon}$ and will be denoted by the ket-vector $|\vec{p}\vec{\epsilon}\rangle$. However, the radiation field has a variable number of photons, and its state must be thus described in the representation of the second quantization, i.e., by the eigenstates

$$|N_1, N_2, \dots\rangle = (N_1!N_2!\dots)^{-1/2} \eta_1^{N_1} \eta_2^{N_2} \dots |0\rangle \quad (3.24)$$

of operators of number of particles

$$\mathcal{N}_j = \eta_j \bar{\eta}_j , \quad (3.25)$$

in the states $|\vec{p}_j \vec{\epsilon}_j\rangle_{j=1}^\infty$. Here η_j , $\bar{\eta}_j$ and \mathcal{N}_j are the creation operator, annihilation operator and the occupation number in the j -th state, resp., and $|0\rangle$ is the vacuum state. The Hamiltonian of a free radiation field reads

$$H_R = \sum_j E_{R,j} \mathcal{N}_j , \quad (3.26)$$

where

$$E_{R,j} = h\nu_j = c|\vec{p}_j| \quad (3.27)$$

is the energy of a photon in the j -th state.

The whole system of the particle plus the radiation field can be described by the ket-vectors

$$|a, N_1, N_2, \dots\rangle = |a\rangle |N_1, N_2, \dots\rangle \quad (3.28)$$

in the product space, and its Hamiltonian is

$$H = H_P + H_Q + H_R , \quad (3.29)$$

where H_Q is the interaction Hamiltonian intermediating the mutual influence between the particle and the radiation field. We can treat it as a small perturbation of the unperturbed Hamiltonian $H_P + H_R$ of a noninteracting particle and field, which has the eigenvectors (3.28). This perturbation will thus cause a transition from an initial state $|i\rangle$ to a final state $|f\rangle$ with the probability per unit time of

$$P_{if} = \frac{4\pi^2}{h} |\langle f|H_Q|i\rangle|^2 \delta(E_i - E_f) \quad (3.30)$$

(E_i, E_f being the energies of the whole system). If the interaction Hamiltonian is expanded into a series in operators $\eta_j, \bar{\eta}_j$,

$$H_Q = V_0 + \sum_j (U_j \eta_j + \bar{U}_j \bar{\eta}_j) + \sum_{jk} (U_{jk} \eta_j \bar{\eta}_k + V_{jk} \eta_j \eta_k + \bar{V}_{jk} \bar{\eta}_j \bar{\eta}_k) + \dots , \quad (3.31)$$

where V_0, U_j, U_{jk}, V_{jk} etc. are some operators in the space of states of the particle (i.e., of kets $|a\rangle$; \bar{U}_j, \bar{V}_{jk} etc. are the corresponding complex conjugates), the individual terms of this expansion can be interpreted as perturbations causing the transition of the particle only (V_0), transition with simultaneous emission or absorption of one photon at the state j (U_j or \bar{U}_j), transition with simultaneous absorption of photon at the state k and emission of another one at the state j (U_{jk}), transition with simultaneous emission of two photons (V_{jk}) and so on.

For instance, the probability of the absorption of a photon at state $|\vec{p}\vec{e}\rangle$ by transition of the particle from the lower state $|l\rangle$ to the upper state $|u\rangle$ can be evaluated according to Eq. (3.30) if we choose

$$|i\rangle = |l, N\rangle \quad \text{and} \quad |f\rangle = |u, N-1\rangle , \quad (3.32)$$

where N is the occupation number of the state $|\vec{p}\vec{e}\rangle$ and the occupation numbers of other states are skipped for brevity. Following the expansion (3.31),

$$\langle f|H_Q|i\rangle = \langle u|\bar{U}|l\rangle \langle N-1|\bar{\eta}|N\rangle = \sqrt{N} \langle l|U|u\rangle , \quad (3.33)$$

and hence according to (3.30) the probability of the absorption is

$$P_{lu} = \frac{4\pi^2}{h} N |\langle l|U|u\rangle|^2 \delta(E_l + h\nu - E_u) . \quad (3.34)$$

In a similar way, to evaluate the probability of emission of a photon by the transition from the upper state $|u\rangle$ to the lower state $|l\rangle$ of the particle, we must choose

$$|i\rangle = |u, N\rangle \quad \text{and} \quad |f\rangle = |l, N+1\rangle . \quad (3.35)$$

Consequently, we find from (3.31),

$$\langle f|H_Q|i\rangle = \langle l|U|u\rangle \langle N+1|\eta|N\rangle = \sqrt{N+1} \langle l|U|u\rangle , \quad (3.36)$$

and the final probability of the emission

$$P_{ul} = \frac{4\pi^2}{h} (N+1) |\langle l|U|u\rangle|^2 \delta(E_l + h\nu - E_u) . \quad (3.37)$$

The polarization of a photon can have two possible values only, while its momentum fills the whole 3-dimensional space. The number N of photons in the state $|\vec{p}\vec{e}\rangle$ is thus equal to the distribution function f of photons, i.e., according to Eq. (2.3), proportional to the specific intensity of the radiation field. Following Eq. (3.34), the probability of the absorption is thus proportional to the intensity of the radiation, which is in agreement with our assumption in the phenomenological description of radiative transfer (cf. Eq. (2.10)), as well as its treatment using Einstein coefficients (cf. Eq. (3.20)). However, the probability of the emission is, following Eq. (3.37), a sum of two terms, one being also linear in the intensity (the induced emission) and the other (the spontaneous emission) being constant. Values of all these terms are determined by the same matrix element $\langle l|U|u\rangle$, which is the quantum-field reason for the validity of Milne – Einstein relations (3.15) and (3.16) justified originally by the thermodynamic reasoning.

For practical calculations of the transition probabilities we must know (in addition to the eigenstates of the atom itself) the interaction Hamiltonian H_Q . For an electron (i.e., a charged particle with a magnetic moment associated with its spin) in an electromagnetic field the interaction Hamiltonian can be expressed as

$$H_Q = \frac{e}{mc}(\vec{p}\vec{A}) + \frac{e^2}{2mc^2}(\vec{A}\vec{A}) - \mu(\vec{\sigma}\vec{H}) , \quad (3.38)$$

where \vec{p} is the momentum, e the charge, m the mass and $\vec{\sigma}$ the spin of the particle, $\mu = \frac{eh}{4\pi mc}$ is the Bohr magneton. \vec{A} is the vector potential of the electromagnetic field, which is the superposition of contributions of individual harmonic modes

$$\vec{A} = \sum_j \sqrt{\frac{hc^2}{2\pi V\nu_j}} \vec{e}_j (\bar{\eta}_j \exp(ik_j x) + \eta_j \exp(-ik_j x)) , \quad (3.39)$$

where k_j is the wave 4-vector, i.e., $k_j x \equiv \frac{2\pi}{h}(\vec{p}_j \vec{x} - E_j t)$. The magnetic field can thus be expressed as

$$\vec{H} = \vec{\nabla} \times \vec{A} = i \sum_j \sqrt{\frac{hc^2}{2\pi V\nu_j}} [k_j \times \vec{e}_j] (\bar{\eta}_j \exp(ik_j x) - \eta_j \exp(-ik_j x)) . \quad (3.40)$$

Substituting (3.39) and (3.40) into (3.38) and comparing the result with (3.31), we find the operator corresponding to the one-photon emission and/or absorption to be

$$U_j = \sqrt{\frac{hc^2}{2\pi V\nu_j}} \left(\frac{e}{mc}(\vec{p}\vec{e}_j) + i\mu(\vec{\sigma} \cdot [k_j \times \vec{e}_j]) \right) \exp(-ik_j x) . \quad (3.41)$$

Following Eq. (3.37), the probability of the spontaneous emission thus reads

$$P_{ul}^{\text{sp.}} = \frac{2\pi c^2}{V\nu_j} |l| \left(\frac{e}{mc}(\vec{p}\vec{e}_j) + i\mu(\vec{\sigma} \cdot [k_j \times \vec{e}_j]) \right) \exp(-ik_j x) |u\rangle|^2 \delta(h\nu + E_l - E_u) . \quad (3.42)$$

Expanding the exponential into the Taylor series ($\exp(-ik_j x) = 1 - ik_j x + \dots$) we get the matrix element of the transition as a sum of dipole, quadrupole and higher-order multipole terms of the electric (the terms with $(\vec{p}\vec{e}_j)$) and magnetic (the terms with $(\vec{\sigma} \cdot [k_j \times \vec{e}_j])$) radiation.

3.3 Thermodynamic equilibrium

It is well known from thermodynamics that the distribution function of any system determines its entropy (related to the amount of information contained in the distribution function), which,

according to the third law of thermodynamics, is non-decreasing function of time. An isolated system thus relaxes to the state of thermodynamic equilibrium for which the corresponding equilibrium distribution function is the maximum of the entropy and is dependent only on the values of conserved quantities of the system (like its energy, momentum, number of particles etc.). Some of them can be eliminated (like the momentum by the choice of the system rest-frame), to others there correspond some ‘intensity’-quantities (like the temperature to the energy or chemical potential to the number of particles), which must have the same value for each (mutually interacting) subsystems of the system in the thermodynamic equilibrium.

An example, which will be of primary importance for our opacity calculations, is the case of non-quantum particles (i.e., particles for which their bosonic or fermionic nature can be neglected, e.g., owing to their small density), for which the thermodynamic equilibrium is described by the Boltzmann distribution

$$f = f_0 \exp\left(-\frac{E}{kT}\right), \quad (3.43)$$

where E is the energy of the state of the particle. The special cases of this distribution are, e.g., the Maxwell distribution of particles velocities (for which $E = \frac{1}{2}mv^2$), the excitation (of electron levels or vibration and rotation modes of molecules), ionization equilibria, or the barometric formula for density of particles in a potential (e.g., the gravity) field ($E = m\Phi$).

Stellar atmosphere is not an isolated system and it is not in thermodynamic equilibrium. However, the relaxation times for reaching the equilibrium are so short for some subsystems of the atmosphere that these subsystem are nearly in equilibrium with some values of the temperature and the other parameters, which are determined by the balance with other subsystems. We thus use to approximate the inner distribution of such subsystems by a state of thermodynamic equilibrium (TE), with corresponding (mutually different) temperatures. This TE is often (nearly) valid for the above mentioned kinetic degrees of freedom in small parts of volume of the atmosphere. In principle, the kinetic temperatures in the same volume element of the atmosphere can be different for different kinds of particles, if they are heated or cooled in a different rate and the exchange of the energy between them is slower than their own Maxwellization (what is the case between the light electrons and heavy ions). However, these differences are mostly negligible. Due to the fast interaction between most (first of all the higher) excitation states of ions with the kinetic degrees of freedom, the distribution of the former tends to the TE with the same temperature and it is thus used to denote the corresponding kinetic temperature as the local temperature of the atmosphere (which is a function of position). If this equilibrium (with the same temperature) is reached by all degrees of freedom, or if we simplify the problem by this assumption, the corresponding state of the atmosphere is denoted as the local thermodynamic equilibrium (LTE). If at least some of the degrees of freedom have different population, their state (and the state of the whole atmosphere) is denoted as non-LTE.

A simple example of LTE is the Boltzmann distribution of electron excitation levels, in which the numerical density of ions at excitation level k is

$$n_k = n \frac{g_k}{Z(T)} \exp\left(-\frac{E_k}{kT}\right) = n \frac{g_k}{U(T)} \exp\left(-\frac{E_k - E_0}{kT}\right), \quad (3.44)$$

where n is the total numerical density of the ion (in any level), g_k is the statistical weight of the level k and

$$U(T) \equiv Z(T) \exp\left(\frac{E_0}{kT}\right) = \sum_k g_k \exp\left(-\frac{E_k - E_0}{kT}\right) \quad (3.45)$$

is the so called partition function.² Let us note, that the partition function diverges for the infinite number of levels. In practice, in the limited volume available for a single atom, only a

²Obviously, U corresponds to populations expressed in terms of the excitation energy $E_k - E_0$, where E_0 is the

finite number of electron orbits can exist due to their dimensions increasing with the excitation energy. Nonetheless, the estimate of the maximum number of levels can non-negligibly influence the calculations of their populations in the case of smaller densities.

Another example of the Boltzmann distribution is the Maxwell distribution of particles into their kinetic degrees of freedom, for which the population of elementary quantum state reads

$$f(x, p) = f_0 \exp\left(-\frac{p^2}{2mkT}\right). \quad (3.46)$$

The normalizing constant f_0 has to be found from the condition for the total number N particles in a reference volume V ,

$$N = nV = \int f(x, p) \frac{d^3 p d^3 x}{h^3} = V \int f_0 \exp\left(-\frac{p^2}{2mkT}\right) \frac{d^3 p}{h^3} = V f_0 \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}. \quad (3.47)$$

We thus see, that its inverse is proportional to the partition function Z_{kin} for the kinetic degrees of freedom,

$$Z_{\text{kin}}(T) \equiv \frac{n}{f_0} = \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}. \quad (3.48)$$

If a particle has both the inner and the kinetic degrees of freedom (or some other in addition), its energy is the sum of the corresponding energies ($E = E^{\text{in}} + E^{\text{kin}} + \dots$) and the equilibrium distribution as well as its overall partition function are products of the corresponding terms for each degree of freedom ($f = f_{\text{in}} f_{\text{kin}} \dots$, $Z = Z_{\text{in}} Z_{\text{kin}} \dots$).

Another important consequence of the Boltzmann distribution in thermodynamic equilibrium is the ionization equilibrium, which is a special case of thermal equilibrium of any chemical reaction



between particles X_i and X_j (cf. Eq. (3.1); here ν , ν' are the stoichiometric coefficients giving the number of particles of the same kind). The left- and the right- hand side of this equation can be treated as two possible composite states of the same physical system. In thermodynamic equilibrium their abundances

$$n_L \equiv \Pi_i (n_i)^{\nu_i} \quad \text{and} \quad n_R \equiv \Pi_j (n'_j)^{\nu'_j} \quad (3.50)$$

(in any possible energetic state of each side of Eq. (3.49)) will thus be proportional to their partition functions

$$n_L = n \frac{Z_L}{Z} \quad \text{and} \quad n_R = n \frac{Z_R}{Z}, \quad (3.51)$$

where $n \equiv n_L + n_R$, $Z \equiv Z_L + Z_R$,

$$Z_L \equiv \sum_{k_i} \exp\left(-\frac{1}{kT} \sum_i E_{i,k_i}\right) = \Pi_i (Z_i)^{\nu_i} \quad \text{and} \quad Z_R = \Pi_j (Z'_j)^{\nu'_j} \quad (3.52)$$

(k_i denotes possible energetic states of the particle X_i). Each particle has the kinetic degree of freedom with the partition function Z_{kin} given by Eq. (3.48), and it can have also inner degrees of freedom with the partition function Z_{in} given by Eq. (3.48), and it can have also inner degrees of energy of the ground level, while Z is referred to any choice of zero energy.

of freedom with Z_{in} given by Eq. (3.45). The equilibrium abundances of individual particles participating in the reaction (3.49) thus satisfy the equation

$$\frac{\Pi_i(n_i)^{\nu_i}}{\Pi_j(n'_j)^{\nu'_j}} = \frac{\Pi_i(Z_i)^{\nu_i}}{\Pi_j(Z'_j)^{\nu'_j}} = \frac{\Pi_i(U_i \exp(-\frac{E_{i0}}{kT})(2\pi h^{-2} m_i kT)^{\frac{3}{2}})^{\nu_i}}{\Pi_j(U_j \exp(-\frac{E_{j0}}{kT})(2\pi h^{-2} m_j kT)^{\frac{3}{2}})^{\nu'_j}}. \quad (3.53)$$

In the special case of the ionization³ of j -times ionized atom X to the ion X_{j+1}

$$X_j \Leftrightarrow X_{j+1} + e, \quad (3.54)$$

We find the Saha equation

$$\frac{n_j}{n_{j+1}n_e} = \frac{U_j \exp(-\frac{E_{j,0}}{kT})}{2U_{j+1} \exp(-\frac{E_{j+1,0}}{kT})(2\pi h^{-2} m'_e kT)^{\frac{3}{2}}} \equiv K_j, \quad (3.55)$$

where $m'_e = m_e m_{j+1}/m_j$ is the reduced mass of the electron. If we know the total density n_X of atoms X (in any ionization state) and the electron density n_e , we can solve (for a given temperature) the set of J linear equations ($j = 0, 1, \dots, J-1$) and the normalization condition

$$\sum_{j=0}^J n_j = n_X, \quad (3.56)$$

for $J+1$ unknown variables n_j . If n_e is also unknown and it has to be determined from the ionization equilibrium at the fixed temperature either for given densities n_{X_k} of all kinds (labeled by k) of atoms (i.e., when the density of the plasma is fixed in addition to its chemical composition) or for a given numerical density n of all free particles

$$n_e + \sum_k n_{X_k} = n \quad (3.57)$$

(i.e., for fixed pressure $P = nkT$ of the gas) we must solve the sets of Saha equations (3.55) and their normalization conditions (3.56) for all k simultaneously with the condition of quasineutrality (the balance of charge)

$$n_e = \sum_{k,j} j n_{k,j} \quad (3.58)$$

either for fixed n_{X_k} in the former case or fixed only the relative abundances $x_k \equiv n_{X_k}/(n - n_e)$ in the later case. For instance, in a pure hydrogen plasma including also the negative ion ($n_{-1} \equiv n_{\text{H}^-}$) at fixed pressure the set of equations reads

$$n_{-1} = n_0 n_e K_{-1}, \quad (3.59)$$

$$n_0 = n_1 n_e K_0, \quad (3.60)$$

$$n_e = n_1 - n_{-1}, \quad (3.61)$$

$$n = n_e + n_1 + n_0 + n_{-1}, \quad (3.62)$$

which yields a quadratic equation

$$n_e^2 K_0 (1 + n K_{-1}) + 2n_e - n = 0, \quad (3.63)$$

³The same results can be obtained both from the set of these subsequent ionizations or from the set of direct ionizations of the neutral atom, $X_0 \rightarrow X_j + je$. It is obvious, that a set of equations of the type (3.49) can be handled as a set of (homogeneous, also negative ν_i are allowed) linear algebraic equations.

with two solutions (one of which is unphysical)

$$n_e = \frac{\pm \sqrt{1 + nK_0(1 + nK_{-1})} - 1}{K_0(1 + nK_{-1})}. \quad (3.64)$$

For a mixture of hydrogen and helium the corresponding equation for n_e is of the 5th degree (or the 4th if H^- is neglected) and its degree increases with the number of ions included. In practice it is thus solved numerically either by straightforward iterations of n_e or by linearization in its perturbations.

Exercise 4 *Investigate the ionization equilibrium of hydrogen plasma with constant density. Find the general equation for n_e , its solution neglecting H^- and its perturbation due to H^- .*

Chapter 4

Model atmospheres and their applications

This chapter contains some selected topics from chapters 7 to 15 of [3] and some additional comments and examples.

4.1 Hydrostatic equilibrium and hydrodynamics

Classical plane-parallel model atmospheres are assumed to be in hydrostatic equilibrium. This assumption is inconsistent for spherical atmospheres. Some observations also confirm that the atmospheres of real stars often continue by a less or more dense outflow of gas called a stellar wind. The hydrostatic equilibrium is a limiting case of the hydrodynamics which has many applications in other fields of astrophysics and physics as well as its technical applications. We will thus derive first the equations of hydrodynamics (starting from a phenomenological insight as well as from the kinetic description of the gas), next we will simplify them for the hydrostatic limit, and finally we will apply them for a spherical stellar wind.

Phenomenological hydrodynamics. The hydrodynamic description of liquids and gasses is usually based on the phenomenological view of a continuum, the physical characteristics of which (like the density ρ , the velocity \vec{v} etc.) are some functions of time t and spatial position described by a chosen ‘Eulerian’ coordinates x (i.e., $f = f(t, x)$). The evolution of the continuum is described by a partial differential equations for these quantities. It is possible to identify individual parts of the continuum and to follow their motion $x = x(t, x_0)$, where x_0 are the ‘co-moving’ ‘Lagrangian’ coordinates. The velocity field is given by

$$v = \frac{\partial}{\partial t} x(t, x_0) , \quad (4.1)$$

and the time-derivatives of all quantities can be rewritten from the Eulerian to the Lagrangian form introducing the ‘flow-’ time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (v \nabla) . \quad (4.2)$$

Choosing some region Ω_0 of Lagrangian coordinates we can find its volume V at the time t as

$$V(t) = \int_{\Omega} d^3x = \int_{\Omega_0} \left| \frac{\partial x(t, x_0)}{\partial x_0} \right| d^3x_0 . \quad (4.3)$$

Its flow- time derivative reads

$$\frac{d}{dt}V(t) = \int_{\Omega_0} \frac{\partial}{\partial t} \left| \frac{\partial x(t, x_0)}{\partial x_0} \right| d^3x_0 = \int_{\Omega_0} \sum_i \frac{\partial v^i}{\partial x^i} \left| \frac{\partial x(t, x_0)}{\partial x_0} \right| d^3x_0 = \int_{\Omega} (\nabla v) d^3x \simeq V \cdot (\nabla v) \quad (4.4)$$

(where we have used the relation $\frac{\partial}{\partial t} \frac{\partial x^i}{\partial x_0^j} = \frac{\partial v^i}{\partial x^k} \frac{\partial x^k}{\partial x_0^j}$ in each term of the derivated determinant; the last equality is valid for an infinitesimally small region Ω).

Let us suppose now, that the total mass $M = \rho V$ of the region is conserved. We thus arrive at the equation of continuity which reads

$$\frac{d\rho}{dt} + \rho(\nabla v) = \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 , \quad (4.5)$$

in the Lagrangian or the Eulerian form, resp.

Similarly, if we suppose that the time- derivative of the momentum of contained in the region is given by the total force acting on the element of continuum (i.e., the pressure acting on its surface as well as the density of force acting in its volume), we arrive at the ‘Navier – Stokes equation’ of motion

$$\rho \frac{dv}{dt} \equiv \rho \left(\frac{\partial v}{\partial t} + (v \nabla) v \right) = f - \nabla P , \quad (4.6)$$

where f is the volume density of the force (e.g., $-\rho \nabla \Phi$ in the case of a scalar potential) and P is the pressure (which can be replaced by an anisotropic stress-tensor for a viscose continuum). The value of the pressure is usually assumed to be determined by an equation of state (e.g., for an ideal gas)

$$P = P(\rho, T) \sim nkT . \quad (4.7)$$

The temperature T , which corresponds to the density of the inner energy (e.g. $\varepsilon \sim \frac{3}{2}nkT$ for the ideal gas), is either given by the thermodynamic equilibrium with some other physical system (e.g., with the radiation field), or it must be solved from the equation of continuity of the energy. Constructing this equation, it must be taken into account that the change of the inner energy of the element of gas is influenced (in addition to other processes with the efficiency $\dot{\varepsilon}$) also by the work exerted by the change of the volume against its pressure, $\frac{d}{dt}(\varepsilon V) = \dot{\varepsilon} - P \frac{dV}{dt}$. For instance, in the case of an ideal gas, this equation reads

$$\frac{d}{dt}\varepsilon + \frac{5}{3}\varepsilon(\nabla v) = \dot{\varepsilon} , \quad (4.8)$$

where on the right-hand side is the volume-density of the other processes like the efficiency of the thermonuclear reactions, the energy absorbed from the radiation (which can be found from the solution of the equation of radiative transfer), or the divergency of the conductive flow of thermal energy.

Exercise 5 (1) *Prove by an explicit calculation in two dimensions that*

$$\frac{\partial}{\partial t} \left| \frac{\partial(x^1, x^2)}{\partial(x_0^1, x_0^2)} \right| = \left(\frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} \right) \left| \frac{\partial(x^1, x^2)}{\partial(x_0^1, x_0^2)} \right| .$$

(2) *Prove the same relation splitting the transform $x_0 \rightarrow x(t)$ into two steps $x_0 \rightarrow x(t') \rightarrow x(t)$ and fixing $t' = t$. (3) Generalize the proof to higher dimensions.*

Hydrodynamic approximation from moments of Boltzmann equation. If we write down Boltzmann equations (2.7) for all kinds of particles of the gas (these particles are generally non-relativistic – unlike the photons treated in Section 2.1) and we calculate their moments in the momentum space, we arrive at partial differential equations for the moments of the distribution functions f . The first three moments are just the quantities: numerical density of particles

$$n = \frac{1}{m} \rho = \int f d^3 p, \quad (4.9)$$

density of their momentum

$$\pi^i = \rho v^i = \int p^i f d^3 p, \quad (4.10)$$

and the stress tensor

$$mT^{ij} = m(\rho v^i v^j + \tau^{ij}) = \int p^i p^j f d^3 p \quad (4.11)$$

(τ is the proper stress tensor with respect to the comoving frame, while T corresponds to the coordinate frame), i.e., the quantities we want to solve from the equations of hydrodynamics. Let us show in the simple case of particles moving in a scalar potential field $\Phi(x)$ that the moment equations are equivalent to the above given equations of continuity, motion etc.

The Hamiltonian of one particle reads

$$H = \frac{p^2}{2m} + m\Phi(x), \quad (4.12)$$

and the Hamiltonian equations of motion

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} = -m \frac{\partial \Phi}{\partial x^i}, \quad (4.13)$$

obviously satisfy the Liouville theorem ($\frac{\partial \dot{x}^i}{\partial x^i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0$). The Boltzmann equation can thus be written in the explicit form

$$\frac{\partial f}{\partial t} + \frac{p_i}{m} \frac{\partial f}{\partial x^i} - m \frac{\partial \Phi}{\partial x^i} \frac{\partial f}{\partial p_i} = \left(\frac{\delta f}{\delta t} \right)_c. \quad (4.14)$$

The zero-order moment (i.e., $\int d^3 p$) of this equation is

$$\frac{\partial n}{\partial t} + \frac{1}{m} \frac{\partial \pi^i}{\partial x^i} = \left(\frac{\delta n}{\delta t} \right)_c, \quad (4.15)$$

where on the right-hand side is the moment of the collisional term, which gives the number density of particles of the particular type created by the collisions (chemical reactions). The integral of the third term on the left-hand side is zero because it is an integral of the derivative of f , which must vanish at infinite p . Substituting here n and p^i from Eqs. (4.9) and (4.10), we arrive, after multiplication by m and summation for all kinds of particles, at the equation of continuity (4.5).

The first moments of Eq. (4.14) can be obtained multiplying it by p^j and integrating it in p ,

$$\frac{\partial \pi^j}{\partial t} + \frac{\partial T^{ji}}{\partial x^i} + \rho \frac{\partial \Phi}{\partial x^j} = \left(\frac{\delta \pi^j}{\delta t} \right)_c, \quad (4.16)$$

where we have used the integration ‘per partes’ $\int p^j \frac{\partial f}{\partial p_i} = -\delta_i^j \int f$. The collisional term on the right-hand side gives the momentum transferred to the particles in collision. It must vanish for a

closed system of particles, but it can give also the force exerted by the radiation pressure, if the radiative processes are included. Using Eq. (4.15), a more common form of the equation of motion of a one-component gas can be obtained (cf. Eq. (4.6))

$$\rho \left(\frac{\partial v^j}{\partial t} + v^i \frac{\partial v^j}{\partial x^i} \right) + \frac{\partial \tau^{ji}}{\partial x^i} + \rho \frac{\partial \Phi}{\partial x^j} = \left(\frac{\delta \pi^j}{\delta t} \right)_c - v^j \left(\frac{\delta \rho}{\delta t} \right)_c . \quad (4.17)$$

In a similar way, the second moments of Eq. (4.14) can be obtained (integrating $\int d^3 p p^j p^k / m$)

$$\frac{\partial T^{jk}}{\partial t} + \frac{\partial Q^{ijk}}{\partial x^i} + \pi_j \frac{\partial \Phi}{\partial x^k} + \pi_k \frac{\partial \Phi}{\partial x^j} = \left(\frac{\delta T^{jk}}{\delta t} \right)_c , \quad (4.18)$$

where on the right-hand side there are contributions of the collisional term to the stress tensor, and Q^{ijk} is the moment of the third order.

In a close correspondence with the moment equations of the radiative transfer, each moment of the Boltzmann equation for the particles of gas contains higher order moments of the distribution function. To get a closed set of equations, it is necessary to estimate them as some functions of lower-order moments (as it is done for moments of radiation field by means of the Eddington factor). This is a matter of different versions of the hydrodynamic approximation. One of the simplest possibilities is to assume the relaxation processes (caused by the collisions between the particles of gas) so effective, that the distribution function f is the equilibrium Maxwellian distribution

$$f = n(2\pi mkT)^{-\frac{3}{2}} \exp \left(-\frac{(p - mv)^2}{2mkT} \right) , \quad (4.19)$$

where the height and the position of the maximum of the Gaussian curve are according to (4.9) and (4.10) given by ρ and v , which must satisfy Eqs. (4.15) and (4.17). Following Eq. (4.11), the stress tensor for the equilibrium distribution (4.19) in the comoving system of the gas and it is proportional to the temperature T of the gas,

$$\tau^{ij} = \delta^{ij} P = \delta^{ij} nkT = \delta^{ij} \frac{2\varepsilon}{3} , \quad (4.20)$$

where P is the pressure and ε is the density of thermal energy. The temperature can be either fixed (e.g., due to the equilibrium with the radiation), or it (or ε) can be found as a solution of the equation of continuity of energy,

$$\left(\frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \right) \varepsilon + \frac{5}{3} \varepsilon \frac{\partial v^i}{\partial x^i} = \left(\frac{\delta \varepsilon}{\delta t} \right)_c , \quad (4.21)$$

which can be obtained from the trace (i.e., $\sum_{j=k}$) of Eq. (4.18), if Q and T are expressed from (4.19) in terms of n, v, T . The second term on the left-hand side gives the above mentioned change of the inner energy due to the work exerted by the change of volume against the pressure of the gas. The collisional term on the right-hand side is either zero for an adiabatic motion of one-component gas, or it can give the exchange of the energy with the other components of gas or the radiation (the later being dominant in the case of the radiative equilibrium). The other components of Eq. (4.18) as well as the higher order moments of the equation yield the criteria of reasonability of the assumption (4.19). If the collisions are not enough efficient to damp the perturbations of the equilibrium distribution arising due to the diffusion of particles in the non-homogeneous or differentially moving gas, the non-diagonal components of the stress-tensor, the heat conduction etc. must be also solved from the higher moment equations including the appropriate expressions

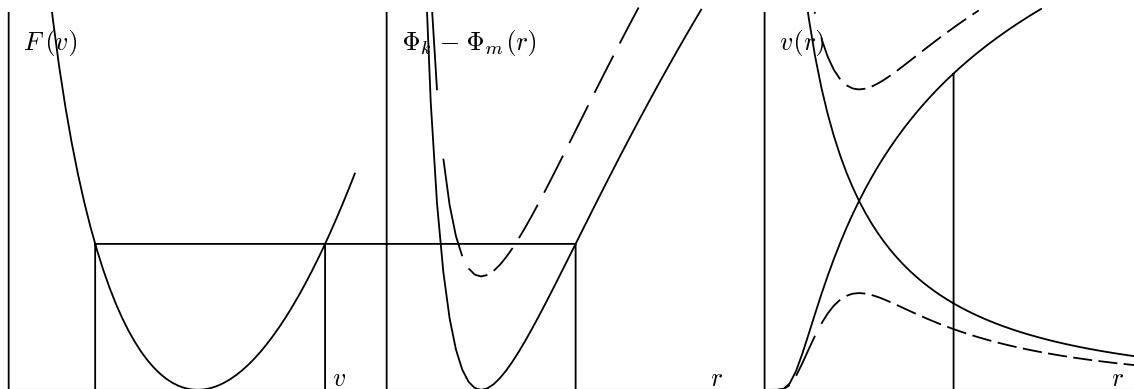


Figure 4.1: Spherically symmetric flow of gas.

for the collisional term. A simple semi-empirical model of the collisional terms can be obtained if the assumption of the exact Maxwellian distribution (4.19) is replaced by the ‘BGK’- model (Bhatnagar, Gross, Krook)

$$\left(\frac{\delta f}{\delta t}\right)_c = -\frac{f - f_0}{\Delta t}, \quad (4.22)$$

where f_0 is the equilibrium distribution with the values of n , v (and facultatively also T) corresponding to the the local distribution f and Δt is a relaxation time. Because the moments of this collisional term are proportional to the difference between the actual moments of f and the moments of f_0 , the former can be expressed as a sum of the later plus a perturbation proportional to the left-hand side of the moment equation. Being the coefficient of the proportionality, the relaxation time Δt thus gives the viscosity, thermal conductivity etc. The main weak point of the approximation (4.22) is that in reality the relaxation is not uniform in the p - space and consequently it can not to fit all these coefficients simultaneously.

Exercise 6 *Derive the moment equations for a gas of charged particles in external electromagnetic field.*

Stellar wind. Let us apply the hydrodynamic equations to a stationary spherically symmetric outflow of gas, which is a model of the so called stellar wind. If we suppose the velocity field to be $v^i = v(r)r^i/r$, we can find the Eulerian equation of continuity (4.5) in the form

$$0 = \nabla^i \left(\frac{\rho(r)v(r)}{r} r^i \right) = r^{-2} \frac{d}{dr} (\rho(r)v(r)r^2), \quad (4.23)$$

which can be integrated as

$$\rho(r)v(r)r^2 = h, \quad (4.24)$$

where the constant $h = -\dot{M}/4\pi$ is the mass flowing per unit time into a unit solid angle (it is positive for the wind outflow or negative in the case of spherically symmetric accretion). The vector equation of motion (4.6) or (4.17) has the only non-zero component in the radial direction

$$\rho(r) \frac{dv}{dr} = -\rho g - \frac{d}{dr} (P + P_r), \quad (4.25)$$

where $g = \frac{d}{dr}\Phi$ is the gravity acceleration and P_r is the radial component of the stress tensor of radiation field (2.6), the gradient of which can be – using the moment (2.28) of the equation of radiative transfer – expressed also by means of the radial flow of the radiation.

According to the equation of state (4.7), the pressure P of the gas is linked to the density ρ and temperature T . If the behavior of the temperature is known, e.g., if it is fixed by the interaction with the radiation field as nearly isothermic, or if it is an adiabatic cooling etc., then the pressure P is determined by the density ρ , which can be, according to (4.24), expressed as a function of v , for which we thus arrived at an ordinary differential equation

$$\left(v - \frac{kT}{m}\right) \frac{dv}{dr} = -g - r^2 \frac{d}{dr} \left(\frac{kT}{mr^2}\right) + \frac{\bar{\kappa}}{c} F_r. \quad (4.26)$$

By its integration we find an implicit algebraic equation for v in the form

$$F(v) = \Phi_k - \Phi_m(r), \quad (4.27)$$

where the function $F(v)$ on the left-hand side reads, e.g., in the isothermic case,

$$F(v) = \frac{1}{2}(v^2 - v_k^2) - \frac{kT}{m} \ln \left| \frac{v}{v_k} \right| \quad (4.28)$$

and it has a minimum equal to zero at a critical point

$$v_k = \sqrt{\frac{kT}{m}}. \quad (4.29)$$

On the right-hand side of Eq. (4.27) there is a modified potential $\Phi_m(r)$, which has in the isothermic case (and for a constant effective opacity per unit mass $\bar{\kappa} = \bar{\alpha}/\rho = \int \alpha F_r d\nu / (\rho \int F_r d\nu)$) the form

$$\Phi_m(r) = \int \left(g - 2\frac{kT}{mr} - \frac{\bar{\kappa}}{c} F_r \right) dr = -\frac{GM}{r} - 2\frac{kT}{m} \ln(r) + \frac{\bar{\kappa}L}{4\pi cr}, \quad (4.30)$$

where G is the gravity constant, M the mass of the star and $L = 4\pi r^2 F_r$ its luminosity. This potential also has a minimum in a critical point

$$r_k = \frac{m}{2kT} \left(GM - \frac{\bar{\kappa}L}{4\pi c} \right). \quad (4.31)$$

To ensure the existence of a smooth solution of $v(r)$ for all r , the integration constants must satisfy the inequality $\Phi_k \geq \Phi_m(r_k)$. If $\Phi_k > \Phi_m(r_k)$, there exist two solutions, one with a subcritical velocity and the other with supercritical velocity (cf. the dashed lines in Fig. 4.1). In the case $\Phi_k = \Phi_m(r_k)$ these solutions are joined in the critical point, and they can be combined into a solution corresponding either to the stellar wind, which is asymptotically static for $r \rightarrow 0$ and dynamical for $r \rightarrow \infty$, or to the radial accretion, for which it is vice versa and the velocity is negative (cf. full lines in Fig. 4.1). There must be satisfied the condition that the luminosity of the star is below the Eddington limit, i.e. $L < \frac{4\pi c}{\bar{\kappa}} GM$, and hence the attractive gravity potential prevails above the repulsive pressure of the radiation.

Chapter 5

Gravitoacoustic waves

5.1 Sound waves in the atmosphere

In this Section, we shall study local behaviour of radial waves, i.e. of small periodic perturbations of hydrostatic equilibrium of a plane-parallel atmosphere. The condition of hydrostatic equilibrium can be obtained from the Navier – Stokes equation (4.6) for a unit volume of fluid in a scalar (gravitational) potential Φ ,

$$\rho \frac{dv}{dt} = -\rho \nabla \Phi - \nabla P, \quad (5.1)$$

if the the velocity v is set equal to zero. The equation of hydrostatic equilibrium thus reads

$$\rho \nabla \Phi + \nabla P = 0. \quad (5.2)$$

Assuming the atmosphere to be an ideal gas, for which the equation of state (4.7) can be written as

$$P = \frac{k}{m} T \rho, \quad (5.3)$$

where m is the mean mass of a free particle (atom, ion, electron), we arrive at a differential equation for the density ρ , if the potential Φ and temperature T are known functions of the radial coordinate r . In particular, for an isothermal atmosphere, $T = \text{constant}$, in homogeneous gravity field, $\Phi = gr$, where the gravity acceleration $g = \text{constant}$, this equation reads

$$\frac{1}{\rho} \nabla \rho = -\frac{mg}{kT}, \quad (5.4)$$

and its solution is the barometric formula

$$\rho = \rho_0 \exp\left(-\frac{mg}{kT} r\right) = \rho_0 \exp\left(-\frac{r}{H}\right), \quad (5.5)$$

where $H \equiv \frac{kT}{mg}$ is the characteristic height scale of the atmosphere.

Let us now investigate a (dynamic) perturbation of the atmosphere, in which a gas element changes its coordinates

$$x \rightarrow x' = x + \delta x \equiv x + \xi(x, t), \quad (5.6)$$

with velocity

$$v = \dot{\xi} \equiv \frac{\partial}{\partial t} \xi(x, t), \quad (5.7)$$

and simultaneously, due to its expansion or compression, it changes also its density and pressure¹

$$\rho \rightarrow \rho' = \rho + \delta\rho, \quad (5.8)$$

$$P \rightarrow P' = P + \delta P. \quad (5.9)$$

The Lagrangian perturbation of the Lagrangian equation of continuity (4.5)

$$\frac{d\rho}{dt} + \rho(\nabla v) = 0 \quad (5.10)$$

reads

$$\frac{d\delta\rho}{dt} + \rho(\nabla\dot{\xi}) = 0, \quad (5.11)$$

and hence, integrating with time, the perturbation of density is determined by the divergence of the displacement²

$$\delta\rho = -\rho(\nabla\xi). \quad (5.12)$$

For fast perturbations, we can assume that the pressure varies according to the adiabatic law

$$P \sim \rho^\gamma, \quad (5.13)$$

where $\gamma = \frac{5}{3}$ for an ideal gas. Consequently, the perturbation of the pressure is related to the perturbation of the density by

$$\delta P = \gamma \frac{P}{\rho} \delta\rho, \quad (5.14)$$

hence, substituting (5.12), we find

$$\delta P = -\gamma P(\nabla\xi). \quad (5.15)$$

The Lagrangian perturbation of the Navier – Stokes equation (5.1) reads

$$\rho\ddot{\xi} = -g\delta\rho - \delta\nabla P. \quad (5.16)$$

Substituting here from (5.12), (5.1), (5.15) and the barometric formula (5.5) for P , we find

$$\begin{aligned} \rho\ddot{\xi} &= g\rho(\nabla\xi) - \delta_E\nabla P - \xi\nabla^2 P = -(\nabla\xi)\nabla P - \nabla\delta_E P - \xi\nabla^2 P = \\ &= -\nabla(\delta_E P + \xi\nabla P) = -\nabla(\delta P) = \gamma\nabla(P\nabla\xi) = \\ &= \gamma P(\nabla^2\xi - \frac{1}{H}\nabla\xi). \end{aligned} \quad (5.17)$$

This is the wave-equation for ξ ,

$$\ddot{\xi} = c^2(\nabla^2\xi - \frac{1}{H}\nabla\xi), \quad (5.18)$$

¹The perturbation δ of any physical quantity f is here the Lagrangian perturbation (i.e. that perceived by the element of fluid), $\delta f \equiv f'(x+\xi) - f(x)$. There can also be introduced an Eulerian perturbation (i.e. that measured at a fixed point x), $\delta_E f \equiv f'(x) - f(x)$. Obviously, $\delta f = f'(x) + \xi\nabla f - f(x) = \delta_E f + \xi\nabla f$ (apart from terms of higher orders of the perturbation) and only the Eulerian perturbation commutes with the partial time or space derivatives. Both perturbations are identical for quantities constant in the unperturbed state (in particular for ξ and v , which are zero in the unperturbed atmospheres and hence $\delta\xi \equiv \xi$ and $\delta v \equiv v$). Note that in literature (e.g. [7]) the Eulerian perturbation is often denoted by δ and the Lagrangian perturbation by Δ , which is consistent with their mathematical equivalence with the synchronous and asynchronous variation, resp. However, [9] and their followers use prime (') for the Eulerian and δ for the Lagrangian perturbation.

²Alternatively, we can start from an Eulerian perturbation of the Eulerian form of the equation of continuity, $\frac{\partial\rho}{\partial t} + \nabla(\rho v) = 0$, and find the solution $\delta_E\rho = -\nabla(\rho\xi)$. Note that Eqs. (7.3) and (7.4) of [5] are thus incorrect.

where the speed of sound is

$$c = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k T}{m}}. \quad (5.19)$$

Unlike the case of a homogeneous gas, the second term on the right-hand side of Eq. (5.18) causes a change of the amplitude of ξ . Because the density of energy (e.g. kinetic energy) of the wave is proportional to $\rho \dot{\xi}^2$, let us rescale the amplitude to another variable $\xi \rightarrow X = \sqrt{\rho} \xi$, the square of which will be directly proportional to the energy density independently of the local value of the density. Substituting this new variable into Eq. (5.18), we find

$$\begin{aligned} \ddot{X} &= c^2 \sqrt{\rho} \left(\nabla - \frac{1}{H} \right) \nabla (X \rho^{-1/2}) = c^2 \sqrt{\rho} \left(\nabla - \frac{1}{H} \right) \left(\nabla X \rho^{-1/2} + \frac{1}{2H} X \rho^{-1/2} \right) = \\ &= c^2 \sqrt{\rho} \left(\nabla^2 X \rho^{-1/2} + \frac{1}{H} \nabla X \rho^{-1/2} + \frac{1}{4H^2} X \rho^{-1/2} - \frac{1}{H} \nabla X \rho^{-1/2} - \frac{1}{2H^2} X \rho^{-1/2} \right) = \\ &= c^2 \left(\nabla^2 X - \frac{1}{4H^2} X \right). \end{aligned} \quad (5.20)$$

Using the Fourier transform

$$X(x, t) = A(k, \omega) e^{i(\omega t - kx)}, \quad (5.21)$$

Eq. (5.20) turns into an algebraic equation for the amplitude A

$$\left(\omega^2 - c^2 k^2 - \frac{c^2}{4H^2} \right) A(k, \omega) = 0. \quad (5.22)$$

This linear homogeneous equation has a non-trivial solution for A , if the frequency satisfies the dispersion relation

$$\omega = \pm \sqrt{c^2 k^2 + \omega_c^2}, \quad (5.23)$$

where the cut-off frequency

$$\omega_c = \frac{c}{2H} = \frac{g}{2} \sqrt{\frac{\gamma m}{kT}}. \quad (5.24)$$

5.2 Gravity waves

In the previous Section, we assumed that the whole horizontal plane in the atmosphere oscillates simultaneously up and down, i.e. the wave-vector has a horizontal component equal to zero. Here, we shall study the opposite limit, in which only individual blobs of gas with small horizontal size will oscillate up and down in their environment, which is in the mean in the hydrostatic equilibrium. The blob can thus expand horizontally to hold its inner pressure in equilibrium with the unperturbed atmosphere. It means that the Eulerian perturbation of pressure is zero and the Lagrangian perturbation reads

$$\delta P = \xi \nabla P. \quad (5.25)$$

We assume that the expansion of the blob is again adiabatic, i.e. Eq. (5.14) is still valid and it determines the variation of density³

$$\delta \rho = \frac{\rho}{\gamma P} \xi \nabla P. \quad (5.26)$$

³Note that Eq. (5.12) is still valid for the vector ξ , and it determines its total divergence both in vertical and horizontal direction.

Generally, the total gravity and buoyancy force acting on the blob of mass M reads

$$f = M \left(-\frac{1}{\rho} \nabla P - \nabla \Phi \right), \quad (5.27)$$

and it is equal to zero in the equilibrium. The perturbation of this force caused by the displacement of the blob⁴ reads

$$\delta f = M \left(\frac{\delta \rho}{\rho^2} \nabla P - \frac{1}{\rho} \delta \nabla P \right). \quad (5.28)$$

Consequently, the blob will move with acceleration

$$\ddot{\xi} = \frac{\delta f}{M} = \frac{1}{\rho} \left(\frac{\delta \rho}{\rho} \nabla P - \delta \nabla P \right) = \frac{1}{\rho} \left(-g \delta \rho - \delta_E \nabla P - (\xi \nabla) \nabla P \right) = -\frac{g}{\rho} \left(\delta \rho - (\xi \nabla) \rho \right), \quad (5.29)$$

where we have used twice Eq. (5.2) and the fact that $\delta_E \nabla P = \nabla \delta_E P = 0$. Substituting here from Eq. (5.26), we arrive at the equation of motion for ξ

$$\ddot{\xi} = -\frac{g}{\rho} \left(\frac{\rho}{\gamma P} \nabla P - \nabla \rho \right) \xi, \quad (5.30)$$

which is the standard equation of a harmonic oscillator with the circular frequency (so called Brunt - Väisälä frequency) N , which can be expressed (using Eq. (5.3)) as

$$N^2 = g \left(\frac{1}{\gamma P} \nabla P - \frac{1}{\rho} \nabla \rho \right) = -g \left(\frac{\gamma - 1}{\gamma P} \nabla P - \frac{1}{T} \nabla T \right). \quad (5.31)$$

A negative value of N^2 (i.e. an imaginary value of N) in a case of large (negative) gradient of T would indicate a convection instability. In the case of the isothermal atmosphere

$$N = \sqrt{\frac{\gamma - 1}{\gamma} \frac{g}{H}} = g \sqrt{\frac{\gamma - 1}{\gamma} \frac{m}{kT}} = \frac{2\sqrt{\gamma - 1}}{\gamma} \omega_c \simeq 0.979796 \omega_c \quad (5.32)$$

for an ideal gas.

In the case of a general wave in the atmosphere, both effects (i.e. the dilution of density and the buoyancy) influence its propagation depending on the horizontal k_h and the radial (vertical) k_r component of the wave-vector. The dispersion relation for these general gravito-acoustic waves then reads

$$(\omega^2 - \omega_c^2) \omega^2 - (k_h^2 + k_r^2) c^2 \omega^2 + k_h^2 c^2 N^2 = 0. \quad (5.33)$$

For $k_h = 0$ this equation reduces to Eq. (5.23). For a general value of $k_h = 0$, there are two regions of ω , for which a non-negative value of k_r^2 may exist. One is the region of high-frequency pressure waves, and the other is of low-frequency gravity waves.

Exercise 7 Find the commutation relations between the Lagrangian perturbation and partial derivatives (with respect to time or coordinate) for a general quantity $f(x)$. Prove that the Lagrangian perturbation commutes with the Lagrangian time derivative $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (v \nabla)$. Cf. [7], Chapter 6.2.

⁴In a planeparallel atmosphere $g = \nabla \Phi = \text{constant}$, so that its variation is zero. However, for a perturbation of a spherical star the corresponding additional term may appear.

Appendix A

Generalised functions

Mathematical theory of generalised functions (distributions) can be found in mathematical textbooks. Here we will remind that they can be introduced as limits of continuous differentiable functions (going sufficiently fast to zero at $\pm\infty$ to guarantee the existence of needed integrals), e.g.,

$$\delta(x) = \lim_{n \rightarrow \infty} \pi^{-\frac{1}{2}} n \exp(-(nx)^2) . \quad (\text{A.1})$$

The convolution $f * g$ of the functions f, g is defined

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy . \quad (\text{A.2})$$

Using the substitution $z = x - y$, it can be verified that

$$f * g = g * f . \quad (\text{A.3})$$

It can be found for derivatives that

$$(f * g)' = f' * g = f * g' , \quad (\text{A.4})$$

or generally

$$(f * g)^{(n+m)} = f^{(n)} * g^{(m)} . \quad (\text{A.5})$$

For the Fourier transform

$$[\mathcal{F}f](y) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixy} dy , \quad (\text{A.6})$$

it can be verified that

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) , \quad (\text{A.7})$$

and vice versa.

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